Supertasks, physics and the Axiom of Infinity

This is a preprint of Chapter 11 of *Truth, Objects, Infinity: New Perspectives on Benacerraf Philosophy*. Ed. by F. Pataut. Springer. 2016

Introduction

It seems reasonable to assume that mathematical infinity was not the objective of Zeno's Dichotomy (in any of its variants), however, a sort of mathematical infinity was already present in these celebrated arguments. Aristotle proposed a first solution to Zeno's Dichotomy by introducing what we now call one-to-one correspondences, the key instrument of modern infinitist mathematics. But Aristotle, more naturalist than platonic, finally rejected the method of pairing the elements of two infinite collections (in this case of points and instants) and introduced instead the distinction between actual and potential infinities. Aristotle's distinction served to define, gross modo, two opposite positions on the nature of infinity for more than twenty centuries. The actual infinity was finally mathematized through set theory in the first years of the XX century and the discussions on its potential or actual nature almost vanished. But, as we will see here, things still remain to be said on this issue.

During the last decades of the 19th century, Bolzano, Dedekind and notably Cantor, inaugurated a new infinitist era in the history of mathematics, which included the birth of set theory. As could not be otherwise, bijections and ellipsis played a capital role in the foundation and subsequent development of the new infinitist theory. Interestingly, set theory was founded on a violation, the violation of the old euclidian Axiom of the Whole and the Part. Indeed, Dedekind's foundational definition states that a set is infinite if it can be put into a one-to-one correspondence with one of its proper subsets. For this reason, Bolzano did not dare to consummate the violation, a task that Dedekind and Cantor finally completed. The success of set theory as the fundamental theory of modern mathematics catapulted set theoretical infinitism to an absolutely hegemonic position. Yet, the controversy surrounding the infinite is not over. Before addressing the heart of the controversy, we need to critically examine the mathematical foundations of contemporary infinitism, and that will be our starting point.

It is somewhat ironic that set theory, the infinitist theory par excellence, contains mathematical instruments that may serve to call into question the formal consistency of the actual infinity hypothesis (the existence of infinite collections as complete totalities). One of those instruments is ω, the first transfinite ordinal, the smallest ordinal greater than all finite ordinals. This first transfinite ordinal defines a type of well-order, called ω-order, that characterizes the most basic infinite objects in transfinite mathematics, as ω-ordered sets and ω-ordered sequences. Most supertasks, for instance, are ω-ordered sequences of actions performed in a finite time interval. Inevitably, ω-order implies a colossal asymmetry largely ignored in infinitist literature. This asymmetry, in turn, gives way to a dichotomy that ultimately results in contradictions, whose ultimate cause can only be the Axiom of Infinity that legitimates ω and ω-order. In this work, we will use a formal version of Zeno Dichotomy to examine this transition from asymmetry to inconsistency via dichotomy.

Two seminal papers published at the beginning of the second half of the 20th century, laid the foundations for a new infinitist theory independent of set theory that has been developing throughout the last decades of the 20th century and the first few years of the 21st. We refer to James Thomson's work [2] on what he called supertasks and to the criticism this work received from Paul Benacerraf [3]. The success of Benacerraf's criticism somehow motivated the subsequent development of the new infinitist theory: supertask theory. Although, also in this case, the controversy is not over. Indeed, supertasks can also be used to question the hypothesis of the actual infinity subsumed into the Axiom of Infinity. This contribution forms part of these critical positions. As we will see later, we continue Benacerraf's arguments at precisely the point he ended his own.

Supertasks are carried out by theoretical artefacts usually known as supermachines or infinite machines. The problem with machines, including theoretical supermachines, is not the (finite or infinite) number of actions to be performed, but the machine's changes of state involved in each performed task. As is well known, the problem of change, another pre-Socratic inheritance, does not have a consistent solution within the space-time continuum. Therefore, dealing with machines that undergo changes of state has the inconvenience of facing an additional problem — the problem of change. We will see what can be done on this issue.

Definitions, procedures and proofs with infinitely successive steps are usual in mathematics. Even though mathematics is neither concerned nor interested in the way these infinitely successive steps could be carried out, the definitions, procedures and proofs of infinitely successive steps can be timetabled and converted into mathematical supertasks. These supertasks have the advantage of not requiring the use of supermachines, theoretical as they may be. Hence, free of the theoretical complications it is possible to argue exclusively in mathematical terms, and then to analyse the consequences of assuming the Axiom of Infinity. Some of these mathematical supertasks will be discussed here.

Although supertasks are also discussed from a physical perspective, our goal here is not to involve physics in supertask theory but to illustrate the way supertasks could be used to call into question the hypothesis of the actual infinity. By the same token, we will also explain why such questioning is of great interest to the experimental sciences such as physics. As a result, we will only focus on conceptual supertasks. Furthermore, all of our arguments will be developed in a conceptual scenario absolutely favourable to the actual infinity hypothesis, without any physical or chemical restriction limiting the discussions. Notwithstanding, we will also take into account physics and infinity, particularly the restrictions that the Planck scale and Planck universal constants impose on supertasks and, what is more interesting, on the infinitist continuums involved in the special theory of relativity.

Platonism is the natural home of infinity and transfinite mathematics. In general, modern mathematics are essentially platonic and a significant number of contemporary mathematicians are also essentially platonic, which is shocking from our perspective of natural sciences. For this reason, we will conclude this work by questioning Platonism (Platonic idealism) from a biological perspective, since we believe that evolutionary biology and neuroscience could shed some light on that classical conception of human knowledge.

The grounds of transfinite mathematics

As said above, Dedekind's foundational definition states that a set is infinite if it can be put into a one-to-one correspondence with one of its proper subsets. It is, therefore, an operational definition of infinite sets based on the violation of Euclid's Axiom of the Whole and the Part. Note that this definition says nothing on the potential or actual nature of the involved infinitude. It is simply taken for granted that the infinity in question is the actual infinity. In other words, it is presupposed that infinite sets are complete totalities. This is due to the fact that potentially infinite sets are not even considered in most mathematical discussions. Bolzano, Dedekind and Cantor unsuccessfully tried to prove the existence of actually infinite sets. Bolzano's proof is as follows (from [3] p. 112):

> One truth is the proposition that Plato was Greek. Call this p1. But then there is another truth p2, namely the proposition that p1 is true [there is then another truth p3, namely the proposition that p2 is true]. And so ad infinitum. Thus the set of truths is infinite.

But this endless process (p1 is true, then p2 is true, then p3 is true, then…) does by no means prove the existence of a final result as a complete totality; Dedekind's proof is similar, and Cantor's one is even less successful ([4], p. 25):

> Each potential infinite presupposes an actual infinity.

It is clear why the existence of an actually infinite set had to be established in axiomatic terms. Precisely, this is the objective of the Axiom of Infinity. In symbols:

$$\exists N(\varnothing \in N \wedge \forall x \in N(x \cup \{x\} \in N))$$

Notice again that the Axiom of Infinity makes no reference to the type of infinitude of the infinite set whose existence is being stated. As in the case of Dedekind's definition, it is supposed that we are talking about the actual infinity.

One-to-one correspondences (bijections or exhaustive injections) are not only present in Dedekind's foundational definition, but throughout the whole history of infinity where they were used in a great variety of arguments, most of them trying to prove (or disprove) the actual infinity hypothesis. They have been and continue to be (along with the inevitable ellipsis) an essential instrument in the development of transfinite mathematics. Here we analyse them at the most basic foundational level of set theory.

It is reasonable to assume that two sets A and B have the same number of elements if it is possible to pair each different element of A with a different element of B, and therefore, that all elements of A and B end up paired (exhaustive injection). But it is also reasonable to assume, and for the same reasons, that if one or more elements of B result unpaired (non-exhaustive injections), then A and B do not have the same number of elements. The existence of both exhaustive and non-exhaustive injections between two infinite sets could indicate they have and do not have the same cardinality. Thus, the arbitrary distinction of the exhaustive injections to the detriment of the non-exhaustive ones could be concealing a fundamental contradiction in set theory. We will begin by analysing this 'apparent' conflict.

If the notion of set is primitive (as it seems to be in the platonic scenario), we need operational definitions in which the pairing method seems to have a basic foundational role. Also, if sets have different sizes (cardinalities), we should establish an appropriate method for comparing cardinalities. We need to do so before defining the types of sets that can be defined according to their cardinals, and before doing any other arithmetic or set theoretical operation. Exhaustive and non-exhaustive injections are the only known basic instruments to accomplish this goal. Therefore, it is at this basic and foundational level of set theory where we need to discuss whether or not the pairing method is appropriate to compare the cardinality of any two sets. If the method is appropriate, then we need to explain why non-exhaustive injections are rejected, since this rejection could be pointing to a fundamental contradiction in set theory: infinite sets have and do not have the same cardinality as some of their proper subsets.

It could be argued that infinite sets are defined as those that can be put into a one-to-one correspondence with one of their proper subsets and that, for this reason, it is possible to define exhaustive and non-exhaustive injections between any infinite set and some of its proper subsets. However, a simple definition does not guarantee that the defined object is consistent. Definitions themselves can also be inconsistent. Furthermore, the existence of a one-to-one correspondence between two infinite sets does not prove that they are actually infinite since both of them could also be potentially infinite. In this latter case, the infinite sets would not be regarded as complete totalities, and thus, we would be pairing the elements of two incomplete totalities of the same cardinality. More importantly, in this condition, the Axiom of the Whole and the Part would not be violated.

Dedekind's definition could be based on one of the terms of a contradiction: the existence of an exhaustive injection between the infinite set and one of its proper subsets. The existence of a non-exhaustive injection between the infinite set and the same proper subset would be the other side of the contradiction. No one has ever explained, except in circular terms, why having an exhaustive injection and a non-exhaustive injection with the same proper subset is not contradictory. The problem has simply been ignored, or justified in behalf of certain properties of infinite cardinals, all of them derived from the foundational definition that is being justified. In our opinion, however, this problem needs to be addressed even before defining what an infinite set could be. Otherwise, set theory would lack a consistent basis.

The arithmetic peculiarities of transfinite cardinals, such as $\aleph_o = \aleph_o + \aleph_o$ and the like, could be used to explain why it is possible to define exhaustive and non-exhaustive injections between a set and one of its proper subsets. However, these arithmetic peculiarities are formal consequences of assuming the existence of sets that can be put into exhaustive and non-exhaustive injections with some of their proper subsets. Therefore, we cannot make use of those arithmetic peculiarities to justify the existence of exhaustive and non-exhaustive injections between a set and one of its proper subsets, otherwise, it will lead us to unacceptable circular reasoning. In short, at this foundational level of set theory, we cannot use posterior attributes of infinites sets derived from the foundational assumptions to justify those foundational assumptions.

If exhaustive and non-exhaustive injections do have the same validity as instruments to compare the cardinality of any two sets, then the actually infinite sets would be inconsistent. If they don't, we should explain, in non-circular terms, why exhaustive injections are valid instruments to compare the cardinality of infinite sets while non-exhaustive injections are not. Recall that both types of correspondences use the same pairing method. And if no (circular) reason can be given, we would have to admit that the position to consider both types of injections are valid instruments to compare cardinalities is as legitimate as the position to consider they are not. Leaving this problem unsolved compels us to declare the arbitrary distinction of exhaustive injections as a new axiomatic fundament of set theory.

Meanwhile, the foundation of set theory may rest on a contradiction. The paradoxes of reflexivity (like Galileo's celebrated paradox) are simple consequences of assuming the existence of exhaustive and non-exhaustive injections between a set and one of its proper subsets. In other word, they are consequences of the violation of the old euclidian Axiom of the Whole and the Part. Clearly, they could also be reinterpreted as contradictions derived from the inconsistent nature of the actually infinite sets (and thus consequences of the Axiom of Infinity). But this alternative, as legitimate as it may be, has always been ignored.

The paradoxes of reflexivity are not the only paradoxes related to infinite sets. Burali-Forti's paradox of the set of all ordinals and Cantor's paradox of the set of all cardinals are other well-known examples. Though in these cases they are not paradoxes but true contradictions. According to Cantor, the inconsistent nature of those sets would be a consequence of their excessive infinitude, too close to the absolute infinity, the mother of all infinities that directly leads to God. It can be proved, however, that the Cantor inconsistency can easily be extended with the aid of Cantor's theorem of the power set. That extension proves (in naïve set theory) that each (finite or infinite) set of cardinal C originates no fewer than 2^C inconsistent sets, all of them infinite. A short for the proof is as follows:

In naïve (non-axiomatic) set theory, the elements of a set can be sets, sets of sets, sets of sets of sets… So it makes sense to define the following relation **R** between two sets A and B: set A is **R**-related to set B (symbolically A **R** B) if B contains at least one element which forms part of the definition of at least one element of A. For instance, the sets:

{{{a, {b}}}, {p}, d, {{{e}}}, f} and {a, b, c} are **R**-related through the elements a and b

{{{a, {b}}}, {c}, d, {{{e}}}, f} and {1, 2, 3} are not **R**-related

In these conditions let X be any nonempty set, and Y any of its subsets, and let us define the following set C_Y of all sets A that are not R-related to any set B that contains elements of Y:

$$C_Y = \{A \mid \neg\exists B(B \cap Y \neq \varnothing \wedge A\ \mathbf{R}\ B)\}$$

If $P(C_Y)$ is the power set of C_Y then any element of $P(C_Y)$ is a subset of C_Y, and then a set of sets that are not R-related to any set that contains elements of Y:

$$\forall D \in P(Y): \neg\exists B(B \cap Y \neq \varnothing \wedge D\ \mathbf{R}\ B)$$

Thus:

$$\forall D \in P(C_Y):\ D \in C_Y$$

And then the cardinal of $P(C_Y)$ is equal or less than the cardinal of C_Y, which contradicts Cantor's theorem of the power set (the cardinal of any set is less than the cardinal of its power set).

Had we known the existence of such an infinitude of inconsistent sets (far less infinite than Cantor or Burali-Forti sets), perhaps transfinite set theory would have had a very different reception. But that was not the case, and for more than half a century all the efforts were directed at establishing a foundation for set theory free of inconsistencies. The goal was finally accomplished with the aid of a considerable number of ad hoc axioms. All them grouped in different ways served to establish at least half a dozen axiomatic set theories. Indeed, we would need several hundred pages to explain all these axiomatic restrictions. One may wonder if this is the best way of founding a formal science. The alternative to dealing with all these axioms is to consider a simpler explanation for all the inconsistencies arising from set theory: that the actual infinity may be inconsistent, an avenue that needs to be explored.

ω-Order: From asymmetry to inconsistency

Cantor's Beiträge [4] (Contributions to the founding of the theory of transfinite numbers) was the last and most mathematical publication of G. Cantor on transfinite arithmetic. In the second paragraph of its epigraph 6 we can read:

> The first example of a transfinite aggregate is given by the totality of finite cardinal numbers v; we call its cardinal number 'Aleph-zero' and denote it by $\aleph_o$.

Cantor took it for granted the existence of that set as a complete totality actually infinite. It is reasonable to suppose that his profound teoplatonic convictions may explain why he did not consider the existence of that complete totality as an initial foundational hypothesis (the Axiom of Infinity in contemporary set theories). In any case, from that infinite totality he successfully derived an infinitude of growing transfinite cardinals and ordinals. In most of these proofs, Cantor made extensive use of his concept of equivalent sets or equipotent sets: sets that can be put into a one-to-one correspondence. This concept also illustrates the great importance of bijections, and the violation of the Axiom of Whole and the Part in the foundation of transfinite mathematics.

Particularly significant is theorem I (Part II, 14) that proves the existence of ordinals as limits of increasing fundamental sequences of ordinals. According to Cantor's terminology, these are the ordinals of the second class, second kind. The first transfinite ordinal, ω, is the first of the second class, second kind transfinite ordinals: it is the smallest of all ordinals greater than all finite ordinals. This ordinal defines a type of well order usually known as ω-order. The set of natural numbers in their natural order of precedence is a well-known example of ω-ordered set. It is important to highlight that ω-order and ω-ordered sequences will play an important role in what follows. For now, let us note that their existence is formally deduced from the Axiom of Infinity.

We now begin our journey from ω-order to ω-inconsistency. The first stage of this journey will be from ω-order to ω-asymmetry. To begin with, consider any ω-ordered sequence a_1, a_2, a_3,... In these type of sequences there is a first element a_1, and each element a_n has an immediate predecessor a_{n-1} (except a_1), and an immediate successor a_{n+1}, so that no last element exists. As a consequence of this type of ordering, every element in the sequence has a finite number of predecessors and an infinite number of successors. We call this asymmetry, ω-asymmetry. Since infinitist mathematics consider ω-ordered sequences as complete totalities, we could travel through each of the successive elements of the sequence and complete the journey even in a finite time. But even if we managed to complete this journey, we will never reach an element with an infinite number of predecessors and a finite number of successors. From the start to the end of this infinitist excursion, we will always be dealing with elements that have a finite number of predecessors and an infinite number of successors. This sort of Red Queen's race to nowhere is known as ω-asymmetry.

To grasp the colossal magnitude of ω-asymmetry, consider a finite straight-line segment AB trillions of times greater than the diameter of the visible universe (9.3×10^{10} light years). Consider also a point C in AB arbitrarily close to B, and let us assume that AB is ω-partitioned. Whatever the ω-partition is, only a finite number of parts will lie within AC, while infinitely many of them will lie within CB, being CB trillions of times smaller than, for instance, Planck length (1.62×10^{-33} cm) that, in turn is inconceivably smaller than, for instance, the smallest of the atomic nuclei. Due to ω-asymmetry, there is no way of performing a less asymmetric partition if the partition is ω-ordered, the smallest of the infinite partitions. And, whatever part you consider (even within CB), it will always have a finite number of preceding parts and an infinite number of succeeding ones. This is how ω-asymmetry works.

Let us now travel from ω-asymmetry to ω-dichotomy. In order to do so, consider the X axis of the euclidian space R^3. Let us assume that its interval (0,1) is partitioned by the sequence $\{z_n\}$ of points defined by:

$$z_n = (2^n - 1) / 2^n, \forall n \in N$$

where N is the set of natural numbers. For well-known historical reasons, the points $\{z_n\}$ will be referred to as Z-points (for Zeno's points). Now, consider a mass point P moving through the X axis from point 2 to point -2 at a finite and uniform velocity v. Now assume that at instant t = 0, P is just on the point 1. At instant 1/v it will be at point 0, which means it has traversed all Z-points (do not forget that they form a complete totality). Let f(t) be the number of Z-points that P has traversed at instant t, for any t within the time interval [0, 1/v]. As a consequence of ω-asymmetry we will have:

$$f(t) = 0 \text{ if } t = 0$$

$$f(t) = \aleph_0 \text{ if } t > 0$$

There is no instant t in [0, 1/v] at which f(t) = n, n being any natural number greater than zero. Otherwise we would have to deal with the existence of the last n elements of a ω-ordered sequence, something that is impossible in ω-order (ω-asymmetry). Keep in mind that f is well defined for every t within the interval [0, 1/v]. It maps the set [0,1/v] onto the set $\{0, \aleph_0\}$. In other words, f defines a dichotomy. So, with respect to the number of traversed Z-points, P can only exhibit two states: the state P(0), at which it has traversed 0 Z-points, and the state $P(\aleph_0)$ at which it has traversed $\aleph_0$ Z-points. Intermediate states P(n) at which P would have traversed a finite number n of Z-points simply do not exist. P will always be either at P(0) or at $P(\aleph_0)$. This is ω-dichotomy, a consequence of ω-asymmetry that, in turn, is a consequence of ω-order that, in turn, is formally derived from the Axiom of Infinity.

Finally, let us travel from ω-dichotomy to ω-inconsistency. First, notice that the points of the interval (0, 1) are densely ordered (between any two of them infinitely many other different points do exist), whereas Z-points are not. Each Z-point has an immediate predecessor (except the first one), and an immediate successor, and no other Z-point exists between any two successive Z-points z_n, z_{n+1}. In addition, a distance $d_n = z_{n+1} - z_n = 1/2^{(n+1)}$ greater than zero always exists between any two successive Z-points z_n, z_{n+1} (ω-separation). Consequently, at any finite velocity they can only be traversed in a successive way, one by one, one at a time, one after the other, and in such a way that it takes a time greater than zero to go from any Z-point to its immediate successor or to its immediate predecessor.

We now know P travels from point 1 to point 0 at a finite and uniform velocity v, so it must become $P(\aleph_0)$ from P(0) as it travels from point 1 to point 0. As a consequence of ω-order, there is no last Z-point to begin the transition $P(0)\rightarrow P(\aleph_0)$. Thus, it will be impossible for us to calculate either the distance P must traverse to become $P(\aleph_0)$ from P(0) or the time it takes P to complete the transition. But the transition takes place even if we cannot describe the way it takes place, since P(0) = 0 and $P(1/v) = \aleph_o$. We will now prove that the transition $P(0)\rightarrow P(\aleph_0)$ must be instantaneous. For this purpose, let t be any real number greater than zero, and assume it takes the transition $P(0)\rightarrow P(\aleph_0)$ a time t. Let t' be any element in the interval (0, t). According to ω-dichotomy and being t' > 0, we will have $P(t') = \aleph_o$. Therefore, at t' the transition $P(0)\rightarrow P(\aleph_0)$ has already been completed. Consequently, that transition lasts a time less than t, which is any real number greater than zero. We must conclude that the transition $P(0)\rightarrow P(\aleph_0)$ lasts a time less than any real number greater than zero, i.e., the transition $P(0)\rightarrow P(\aleph_0)$ lasts a null time: it can only be instantaneous. It is worth noting that we are not dealing with a question of indeterminacy derived from the fact that we cannot measure the duration of the transition, but with an impossibility directly derived from ω-dichotomy: the transition $P(0)\rightarrow P(\aleph_0)$ lasts a time less than any real number greater than zero, and this is possible only if it takes a null time.

We have just proved that the transition $P(0)\rightarrow P(\aleph_0)$ must be instantaneous. This implies that P must traverse $\aleph_o$ successive Z-points instantaneously at a finite velocity v. But this is impossible because between any two successive Z-points there is a distance greater than zero (ω-separation). And traversing a distance greater than zero at a finite velocity always takes a time greater than zero. Therefore, it is impossible for P to traverse any two successive Z-points instantaneously. Simply put, at its finite velocity v, Z-points cannot be traversed simultaneously, they can only be traversed successively, one after the other and so that a time greater than zero always elapses between passing over the one and passing over the other. Oddly enough, we must conclude that the transition $P(0)\rightarrow P(\aleph_0)$ must be instantaneous but cannot be instantaneous. In short:

> The transition $P(0)\rightarrow P(\aleph_0)$ takes place.
>
> Due to ω-dichotomy, $P(0)\rightarrow P(\aleph_0)$ can only be instantaneous.
>
> Due to ω-separation, $P(0)\rightarrow P(\aleph_0)$ cannot be instantaneous at a finite velocity.

In addition to the Z-points, we could also consider the Z*-points $\{z_i^*\}$ defined within (0, 1) by:

$$z_n^* = 1 / 2^n, \forall n \in \mathbb{N}$$

Being now f(t) the number of Z*-points to be traversed by P at any instant t within [0, 1/v], f defines a new ω-dichotomy: with respect to the number of Z*-points to be traversed, P can only exhibit two states, the state $P(\aleph_o)$, in which $\aleph_o$ points to be traversed still remain, and the state P(0) in which no Z*-point remains to be traversed. Intermediate states P(n) at which only a finite number of Z*-points would remain to be traversed are impossible. An argument similar to the above argument on the Z-points leads to the following conclusions on the transition $P(\aleph_0) \rightarrow P(0)$:

> The transition $P(\aleph_0) \rightarrow P(0)$ takes place.
>
> Due to ω-dichotomy, $P(\aleph_0) \rightarrow P(0)$ can only be instantaneous.
>
> Due to ω-separation, $P(\aleph_0) \rightarrow P(0)$ cannot be instantaneous at a finite velocity.

This is ω-inconsistency, an almost direct consequence of ω-dichotomy, in turn, a formal consequence of ω-asymmetry (the existence of complete totalities in which each and every element has a finite number of predecessors and an infinite number of successors), in turn, an immediate consequence of the ω-order derived from ω. Recall that ω is the least transfinite ordinal of the second class, second kind whose existence Cantor deduced from assuming the existence of the complete infinite totality of the finite cardinals (the Axiom of Infinity in modern terms).

For exactly the same reasons as in transitions $P(0) \rightarrow P(\aleph_0)$ and $P(\aleph_0) \rightarrow P(0)$, ω-inconsistency will also appear in any ω-ordered sequence of actions $\{a_i\}$ successively performed at each of the successive instants of $\{t_i\}$. This means that an infinite number of those actions would have to be carried out instantaneously, while they can only be carried out in a successive way, and so that a time $\Delta_n t = t_{n+1} - t_n$ greater than zero always passes between any two of those successive actions. To derive a contradiction from an axiom should be a sufficient reason to consider the possibility that the axiom may be inconsistent. But, for unknown reasons, if that axiom is the Axiom of Infinity, then it is not enough. Let us, therefore, continue to examine this issue.

Benacerraf-Thomson: A seminal discussion

The concept of supertask was already implicit in many classical discussions in which infinity was somehow involved, e.g., in Zeno's Dichotomy, as well as in its subsequent Aristotle's criticism. In the XIV century, scholastic Gregory of Remini detailed how infinitely many successive actions could be carried out in a finite time ([3], p. 53):

> If God can endlessly add a cubic foot to a stone – which He can – then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and son on ad infinitum. He would then have before Him an infinite stone at the end of the hour.

But it was at the beginning of the 1950's when these types of discussions became popular, at least in the academic world. In the first place, we find Black's machines [5], which were intended to prove the impossibility of performing an infinite number of successive actions. Black's arguments were then discussed by R. Taylor [6] and J. Watling [7]. Thomson's 1954 paper was precisely motivated by those discussions. In his paper, Thomson introduced the term supertask, and developed several arguments trying to prove the impossibility of such supertasks. Among these arguments, we find the one about his famous lamp that we will examine in the next section. P. Benacerraf successfully criticized Thomson's arguments in a seminal paper published in 1962, and this somehow gave rise to the birth of a new infinitist theory in the last decades of the XX century: supertask theory.

Most of supertasks are ω-supertasks, i.e. ω-ordered sequences of successive actions (tasks) performed at the successive instants of a strictly increasing ω-ordered sequence of instants within a finite interval of time. Here we will only focus on conceptual ω-supertasks. We will assume that all of them are performed along the same strictly increasing and ω-ordered sequence $\{t_i\}$ of instants within the same finite interval of time $[t_a, t_b]$, being each action a_i carried out at the precise instant t_i, and being t_b the mathematical limit of the sequence $\{t_i\}$. They will be denoted by $\{a_i, t_i\}$, $\{b_i, t_i\}$, $\{c_i, t_i\}$, etc.

The possibilities to perform an uncountable infinitude of successive actions were examined, and ruled out, by P. Clark and S. Read [8]. The proof was based on a Cantor's proof on the impossibility of non-countable partitions in the real line [10]. Let us mention that indeed Cantor's proof is not an independent proof but an immediate corollary of his theorem on the countable nature of the set of rational numbers (see below). Supertasks have also been examined from the perspective of non-standard analysis. But, as far as we know, the possibilities to perform hypertasks along hyperreal intervals of time have not been discussed, despite the fact that finite hyperreal intervals can also be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions).

As indicated before, only conceptual ω-supertasks will be dealt with here. And we will begin by briefly recalling one of the supertasks that James Thomson proposed in 1954, the one performed by his famous lamp. We will also recall the criticism this argument received in 1962 from Paul Benacerraf, which hits the nail on the head when it comes to super task discussions. So, as Thomson did in 1954, let us consider one of these supertasks:

> … reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off.

To avoid unnecessary discussions we will complete Thomson's lamp definition with the following constraints:

1. Thomson's lamp has only two states: on and off.
2. The only way of changing the state of Thomson's lamp is by pressing the button.
3. Each change of state takes place at a precise and definite instant.
4. The button pressing and the corresponding lamp's change of state are instantaneous and simultaneous events.

Most variants of Thomson's lamp have been proposed to discuss the possibility of performing a Thomson's supertask in physical terms. Here, our Thomson's lamp will be a theoretical device intended to examine the formal consistency of the Axiom of Infinity.

Before beginning, we must acknowledge that the problem of change is involved in the discussions on supertasks carried out by supermachines that undergo changes of state. As we know, any canonical change from state A to state B (without intermediate states) poses a problem still unsolved in the space-time continuum, where all solutions have been tried out. As has been claimed for a long time ago, canonical changes could be inconsistent. In fact, if the change has a duration greater than zero, the changing object can only be in an unknown state (different from A and B) while the change takes place. This, in turn, poses the problem of change in terms of a new change between the state A and that unknown state, and so forth. On the other hand, if the change is instantaneous it cannot take place in the space-time continuum, since in this continuum no instant has an immediate successor, and a time greater than zero always elapses between any two instants of this continuum. Things could be quite different in discrete spacetimes, where immediate successiveness is an essential characteristic of both space and time. But for now let us focus on what happens in the space-time continuum.

Having recognized that the problem of change is present, we will ignore it for the sake of the discussion. We will assume that change is instantaneous, and discuss supertasks from the perspective of the Axiom of Infinity, focusing our attention on ω-ordering as well as on the corresponding ω-asymmetries, ω-dichotomies and ω-inconsistencies. Bear in mind that, in theory, ω-dichotomies and ω-inconsistencies have nothing to do with the problem of change, except for the fact that infinite sequences exist as complete totalities.

Let $\{c_i, t_i\}$ be Thomson's supertask and assume that each click c_i is performed at the precise instant t_i of the strictly increasing sequence $\{t_i\}$ of instants within the finite interval (t_a, t_b), being t_b the limit of $\{t_i\}$. Let us now summarize Thomson-Benacerraf discussion with the following words by J. Thomson:

> … [The lamp] cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned off without at once turning it on. But the lamp must be either on or off. This is a contradiction.

And by P. Benacerraf:

> The only reasons Thomson gives for supposing that his lamp will not be off at t_b are ones which hold only for times before t_b. The explanation is quite simply that Thomson's instructions do not cover the state of the lamp at t_b, although they do tell us what will be its state at every instant between t_a and t_b (including t_a). Certainly, the lamp must be on or off (provided that it hasn't gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which it is to be The arguments to the effect that it can't be either just have no bearing on the case. To suppose that they do is to suppose that a description of the physical state of the lamp at t_b (with respect to the property of being on or off) is a logical consequence of a description of its state (with respect to the same property) at times prior to t_b.
>
> (Note: t_a and t_b appears respectively as t_0 and t_1 in Benacerraf's paper).

We agree with Benacerraf's argument in that we cannot deduce the state of the lamp at t_b from the sequence of changes of state that the lamp has previously undergone. We also assume that certain properties of the sequence of changes that hold while the number of changes is finite may be not satisfied if that number is infinite. But, as we will see, Benacerraf's argument does not end the discussion. 'Certainly, the lamp must be on or off… but nothing we are told implies which is to be', says Benacerraf. This conclusion will be the starting point of our extension of Benacerraf's argument on $\{c_i, t_i\}$.

Ignoring what the state of a machine is after the machine has performed a supertask does by no mean implies that the machine is not the same machine it was before and while performing the supertask. By the way, the machine could only change its (theoretical or physical) nature after completing the supertask, otherwise, it would be impossible for the machine to complete the supertask. There is no reason, (except reasons of arbitrary convenience) to assume that if we define a theoretical machine to perform a conceptual supertask, after performing the supertask the machine is no longer the same machine it was defined to be – regardless of its current state. Simply put, ignoring the state of the machine is not the same as ignoring the nature of the machine (its formal definition in our conceptual scenario).

Consequently, we presuppose that after performing a supertask, the conceptual objects that participated in the supertask (regardless of their current states) continue to be the same objects they were before and during the execution of the supertask. For example, if x is a rational variable and we redefine it a certain (finite or infinite) number of successive times, we believe that after these redefinitions have been carried out, x will still be a rational variable, and not a red hat or a neutron star. By the same token, if T is a table of real numbers within (0, 1) whose rows are permuted any finite or infinite number of times, then after the permutation has been carried out, T will continue to be a table with the same real numbers it had before its rows were permuted. So, unnecessary as it may seem, we will begin by assuming the following hypothesis:

> H0: The definition of a conceptual object does not change as a consequence of performing any finite or infinite sequence of successive actions with that object.

More specifically, we assume that formal definitions, laws, conditions and constraints are never arbitrarily violated as a consequence of having performed any finite or infinite number of actions. Denying H0, on the other hand, would have catastrophic consequences on transfinite mathematics. For instance, after performing a recursive definition (or procedure, or proof) of infinitely many successive steps (that could also be scheduled in the form of a supertask), we can say nothing on the defined object. And this would also happen to all the axioms, definitions and theorems involved.

If nothing can be said on a conceptual object after having performed a supertask, then nothing can be said either on any mathematical object or result obtained through a sequence of infinitely many successive steps. Evidently, in these conditions, transfinite mathematics would remain empty of content (see mathematical supertasks below). As we will see, we could also define conditional supertasks (particularly, mathematical supertasks) in such a way that each task would be performed only if certain conditions are satisfied. In this case, it would be impossible for us to know whether the number of performed tasks is finite or infinite. Even if we accept that the state of a conceptual object cannot be deduced from its previous states while performing a supertask, according to H0 its formal definition does not change as a consequence of such a performance.

Thomson's lamp revisited

According to H0, Thomson's lamp will be the same Thomson lamp before, during and after performing the supertask $\{c_i, t_i\}$. Consequently, at t_b, after performing $\{c_i, t_i\}$, the lamp will be at a certain state S_b. We are not interested in knowing whether the lamp is on or off at S_b, although by definition a Thomson lamp can only be either on or off. Some authors have claimed it could be in any exotic state different from these two states. But we must insist that if the lamp can be in any exotic state other than on or off, then it is not, by definition, a Thomson lamp. We know that the state of the lamp is S_b at t_b, and it is immediate to prove that at any instant prior to t_b it is impossible for the lamp to have reached S_b, — whatever the state is. In effect, let t be any instant prior to t_b. Since t_b is the limit of the sequence $\{t_i\}$, there will be a t_v in $\{t_i\}$ such that: $t_v \leq t < t_{v+1}$, which means that at t only a finite number v of clicks have been carried out (ω-asymmetry). In consequence, S_b cannot be originated at t for any t within (t_a, t_b). Therefore, with S_b being the state of the lamp at t_b, the state S_b can only originate at the precise instant t_b. Notice that this conclusion is a direct consequence of the fact that t_b is the mathematical limit of $\{t_i\}$ and of the assumption that $\{c_i, t_i\}$ has been carried out along the successive instants of the strictly increasing sequence $\{t_i\}$.

Notice also that while t_b is the mathematical limit of the strictly increasing and upper bounded sequence of real numbers (successive instants) $\{t_i\}$, the state S_b is not the mathematical limit of the sequence of states $\{S_i\}$ = on, off, on, off, on off…, which the lamp undergoes as a consequence of $\{c_i, t_i\}$. Recall that oscillating sequences do not have a limit. Therefore, S_b is the state of a Thomson lamp that originates at a certain and precise instant t_b, otherwise the supertask would not have been completed. And this is all that can be said on S_b and the supertask $\{c_i, t_i\}$.

Now according to the above definition, a Thomson lamp only changes its state by clicking its button. Hence, it is not acceptable to claim that the lamp may change its state by reasons unknown. Remember again that according to H0 a lamp that changes its state by reasons unknown is not, by definition, a Thomson lamp. With t_b being the precise and definite instant at which S_b originates, the button of the lamp had to be clicked at t_b (the clicking and the corresponding change of state are instantaneous and simultaneous events that take place at a precise and definite instant). Yet this is impossible because at t_b the supertask $\{c_i, t_i\}$ has already finished. Thus, t_b is the first instant after performing $\{c_i, t_i\}$, and the button of the lamp has not been clicked at t_b.

Let f(t) now be the number of clicks to be performed at the precise instant t within the closed interval $[t_a, t_b]$. As a consequence of ω-order and ω-asymmetry, each c_i of $\{c_i\}$ has infinitely many successors. Consequently, we will have:

$f(t) = \aleph_0$ if $t < t_b$

$f(t) = 0$ if $t = t_b$

which means that for any natural number n there is no an instant t at which f(t) = n

Otherwise, the impossible last n elements of an ω-ordered sequence would exist, or simply put, an element of the sequence with a finite number n of successors would exist. Therefore, with respect to the number of clicks to be performed, a Thomson lamp can only have two states:

$TL(\aleph_0)$ at which $\aleph_0$ clicks still have to be carried out,

TL(0) at which no click remains to be carried out.

An argument similar to the one above on Z*-points proves that the transition $TL(\aleph_0) \rightarrow TL(0)$ can only be instantaneous, and hence, $\aleph_0$ clicks have to be performed simultaneously. This contradicts the fact that the button of the lamp is clicked successively and that each click c_i happens at the instant t_i, in such a way that a time $\Delta_i t = t_{i+1} - t_i$ greater than zero always elapses between any two successive clicks c_i and c_{i+1} (ω-separation).

To better illustrate the problem posed by the above ω-dichotomy of $\{c_i, t_i\}$ consider a box BX containing a denumerable sequence $\{b_i\}$ of labelled balls $b_1, b_2, b_3, \ldots$, and assume that we remove the balls from the box one by one, in such a way that at each click c_i, (i.e. at each instant t_i) we remove the ball b_i from the box. At t_b all balls will have been removed from BX, exactly as the one-to-one correspondence $g(t_i) = b_i$ proves. If $f(t)$ is now the number of balls to be removed at instant t, we get the same ω-dichotomy of Thomson's lamp we saw above. So, despite the fact that all balls are removed one by one, one after the other, and in such a way that a time $\Delta t = t_{i+1} - t_i > 0$ always passes between the extractions of any two successive balls b_i, b_{i+1} (ω-separation), the box BX will never contain a finite number of balls. BX is emptied by successively removing *one by one* all of the balls but it will never contain …5, 4, 3, 2, 1, 0 balls. The number of balls within the box will always be either $\aleph_0$ or 0. The number of balls inside the box will suddenly change from $\aleph_0$ to 0.

Furthermore, as in the case of S_b (and for the same reasons), the change can only be instantaneous. Thus, an infinite number of balls would have to be removed from the box simultaneously, which is not compatible with the fact that all of them are removed successively, as the bijection $f(t_i) = b_i$ proves. All things considered, one may wonder if is this a new infinitist extravagancy or simply an inconsistency? We can imagine Ockham's opinion. Notice that we are not subtracting cardinals but removing balls from a box under the restriction of a dichotomy formally derived from ω-order, and thus from the Axiom of Infinity. By the way, subtracting cardinals is really a suspicious transfinite arithmetic operation: sometimes it is permitted (Tarski-Bernstein theorem, Tarski-Sierpinski theorem etc.), sometimes it is not; sometime it is consistent, sometimes it is not (Faticoni argument and the like).

For illustrative purposes only, and without going into further details, we will now formalize Benacerraf-Thomson discussion. For this consider the following expressions and their corresponding symbolic representations:

Thomson's lamp on at instant t: $*[t]$

Thomson's lamp off at instant t: $o[t]$

Thomson's lamp on along the interval (t_a, t_b): $*(t_a, t_b)$

Thomson's lamp off along the interval (t_a, t_b): $o(t_a, t_b)$

Click at instant t the lamp being previously on: $c\{[t], *\}$

Click at instant t the lamp being previously off: $c\{[t], o\}$

Click at least one time along the interval (t_a, t_b) the lamp being previously on: $c\{(t_a, t_b), *\}$

Click at least one time along the interval (t_a, t_b) the lamp being previously off: $c\{(t_a, t_b), o\}$

Note the word 'previously' in the expression 'the lamp being previously on (off)', and recall that in the spacetime continuum no instant has an immediate preceding (or succeeding) instant, in the same way that, for instance, the natural number 5 has an immediate predecessor (the natural number 4), or an immediate successor (the number 6). As noted above, this is why the problem of change remains unsolved in the spacetime continuum. And this is a problem that affects all changes we can think of, whether theoretical or experimental. Therefore, we need to leave the problem of change aside, if we want to continue discussing change, including the state changes of a Thomson lamp.

According to the above symbolism, we can formalize some fundamental laws of Thomson's lamp, for instance the following axioms (definition of the lamp):

$$c\{[t], o\} \Rightarrow *[t]$$

$$c\{[t], *\} \Rightarrow o[t]$$

$$*[t] \vee o[t]$$

$$\neg (*[t] \wedge o[t])$$

And the following derived laws:

$$c\{(t_a, t_b), o\} \Rightarrow \exists t \in (t_a, t_b): *[t]$$

$$c\{(t_a, t_b), *\} \Rightarrow \neg *(t_a, t_b)$$

$$o[t_b] \Rightarrow \neg *[t_b, \infty)$$

etc.

We are interested in the following two laws:

BT1: $c\{(-\infty, t_b),*\} \wedge *[t_b, \infty) \Rightarrow \exists\, t \leq t_b: c\{[t], o\} \wedge \neg\, c\{(t, \infty), *\}$

BT2: $c\{(-\infty, t_b),o\} \wedge o[t_b, \infty) \Rightarrow \exists\, t \leq t_b: c\{[t], *\} \wedge \neg\, c\{(t, \infty), o\}$

BT1 reads: if the lamp's button has been clicked at least once within the interval $(-\infty, t_b)$, the lamp being previously on, and the lamp stays on from t_b, then there is an instant t equal or prior to t_b such that the button is clicked at t, the lamp being previously off, and the button is no longer clicked from t. BT2 reads equal except we have to replace on with off and vice versa. Now let us prove BT1 (the proof of BT2 is similar).

H1: Assume that $\neg\exists t \leq t_b: c\{[t], o\}$. We will have:

$$\neg\, c\{(-\infty, t_b], o\}$$

On the other hand, according to the antecedent of BT1 we have:

$$c\{(-\infty, t_b), *\} \Rightarrow \exists\, t < t_b: c\{[t], *\}$$

which means o[t].

From:

$\neg\, c\{(-\infty, t_b], o\}$ and o[t], being $t < t_b$

we derive $o[t_b]$ and then $\neg *[t_b, \infty)$, which goes against the second term of BT1 antecedent. Therefore, if that antecedent is true then H1 is false.

H2: Assume that: $\neg\, \exists\, t \leq t_b: \neg\, c\{(t, \infty), *\}$.

We will have:

$c\{[t_b, \infty), *\}$

which goes against the second term $*[t_b, \infty)$ of BT1 antecedent. So, if this antecedent is true then H2 must be false. The falsehood of H1 and H2 proves BT1. Notice that BT1 is not derived *a la* Thomson, from the successively performed clicks. BT1 is a law directly derived from the laws that define Thomson's lamp. Therefore, if we assume H0, BT1 must hold before, during and after performing any (finite or infinite) number of clicks.

Consider again the supertask $\{c_i, t_i\}$. Assume that the state S_b is on (a similar argument can be developed if it were off although with BT2 in the place of BT1). In these conditions, the antecedent of BT1 would be true. Therefore, its consequent would also be true. However, it is false. In fact, on the one hand, if $t < t_b$, and with t_b being the limit of the sequence $\{t_i\}$, there would exist a t_v in $\{t_i\}$ such that $t_v \leq t < t_{v+1}$ and hence only a finite number v of clicks would have been carried out. On the other hand, t cannot be t_b either, because at t_b the button of the lamp has not been clicked. Consequently, t cannot be an element of $(\leftarrow, t_b]$. Therefore, to perform $\{c_i, t_i\}$ implies the violation of BT1, which goes against H0.

On marbles and boxes

Sometimes we call a paradox what is really an inconsistency. This is the case of the above-mentioned Burali-Forti's and Cantor's paradoxes, among others. Ross' paradox could also belong to this category. It is a supertask in which there is no general agreement, which we will come to a little later. But before doing that, we will introduce expofactorial and n-expofactorial numbers that will be used later to define two variants of Ross' supertask (expofactorials were also introduced independently by C. Pickover [9]). Although we do not need those numbers to define the supertask, they make it easier to address the subject in an appropriate way, while at the same time illustrating how finite natural numbers can be so large that they may prove repulsive.

The expofactorial $n^!$ (note the factorial symbol ! appears as an exponent) of a number n is the factorial n! raised to a power tower of order n! of the same exponent n! Thus the expofactorial of 2 is:

$$2^! = 2! \wedge 2! \wedge 2! = 2 \wedge 2 \wedge 2 = 2^4 = 16$$

But if we try to calculate the expofactorial of 3, in symbols $3^!$, we simply cannot!

$$3^! = 3! \wedge 3! \wedge 3! \wedge 3! \wedge 3! \wedge 3!$$

$$= 6 \wedge 6 \wedge 6 \wedge 6 \wedge 6 \wedge 6$$

$$= 6 \wedge 6 \wedge 6 \wedge 6^{46656}$$

$$= 6 \wedge 6 \wedge 6^{26591197721532267796824894043879185949053422002699243006604327894970735598738829091213422929\ldots}$$

where the incomplete exponent of the last expression of $3^!$ has nothing less than 36306 digits, roughly ten pages of a standard text like this one. We have not been able to calculate the next step even with the aid of a big integer supercalculator. And there still remain three steps to go. So $3^!$ is such a large number that we could certainly not calculate its precise sequence of ciphers (it is not an anodyne sequence of zeroes) even with the aid of the most powerful current computers. Imagine $9^!$ Let alone $100^!$.

Expofactorials are minuscule compared to n-expofactorials, recursively defined from expofactorials: The 2-expofactorial of a number n, denoted $n^{!2}$, is the expofactorial $n^!$ raised to a power tower of order $n^!$ of the same exponent $n^!$

$$n^{!2} = n^! \wedge n^! \wedge \ldots(n^! \text{ times})\ldots \wedge n^!$$

The 3-expofactorial of a number n, denoted $n^{!3}$, is the 2-expofactorial $n^{!2}$ raised to a power tower of order $n^{!2}$ of the same exponent $n^{!2}$

$$n^{!3} = n^{!2} \wedge n^{!2} \wedge\ldots(n^{!2} \text{ times})\ldots \wedge n^{!2}$$

The 4-expofactorial of a number n, denoted $n^{!4}$, is the 3-expofactorial $n^{!3}$ raised to a power tower of order $n^{!3}$ of the same exponent $n^{!3}$

$$n^{!4} = n^{!3} \wedge n^{!3} \wedge\ldots(n^{!3} \text{ times})\ldots \wedge n^{!3}$$

and so on and on. These numbers really are far beyond human imagination. Three arithmetic symbols, $9^{!9}$, suffice to define a number (9-expofactorial of 9) so large that the standard writing of its precise sequence of figures would surely require a volume of paper trillions of times greater than the volume of the visible universe. As we noted above, they are so large that they prove repulsive. But, being finite, they are much smaller than the smallest of the infinite cardinals $\aleph_o$ (or than the improper real number ∞). In some of the following discussions we will make use of the number $9^{!9}$, which for the sake of simplicity will be denoted by the letter H (for huge).

Consider now the following supertask, our first variant of Ross' supertask: At each instant t_i of $\{t_i\}$ we add H marbles (i.e. $9^{!9}$) to an initially empty box A. On the other hand, if the index i is an integer multiple of H (i.e t_H, t_{2H}, t_{3H},…) then one marble is added to another initially empty box B. At t_b, once completed the supertask, A and B will contain the same number of marbles: $\aleph_o$. From the transfinite arithmetic perspective, there is nothing remarkable in this conclusion because transfinite cardinals satisfy things such as $\aleph_o = (\aleph_o)^H$ and the like. But a certain intellectual dissatisfaction is also inevitable.

On the one hand, $\aleph_o$ is the least transfinite number greater than all finite integers. In this sense, it is the upper limit of any strictly increasing ω-ordered sequence of natural numbers. Since the number of marbles in each box forms an strictly increasing ω-ordered sequence of natural numbers, at t_b both boxes contains the same number $\aleph_o$ of marbles. This is fine, but there is a third strictly increasing and ω-ordered sequence of natural numbers $\{d_i\}$, namely, the difference in the number of marbles within A and B as the supertask progresses:

$$\{d_i\} = H, 2H, 3H,\ldots H^2 - 1, H^2 + H - 1, H^2 + 2H - 1, H^2 + 3H - 1,\ldots H^3 - 2\ldots$$

$$d_i = H^{1+a_i} + b_i H - a_i$$

where $a_i = \text{Int}(i/H)$ and $b_i = (i \bmod H)$.

So, if at t_b the number of marbles in A is the limit $\aleph_o$ of the sequence:

$$\{iH\} = H, 2H, 3H,\ldots$$

and the number of marbles in B at the same instant t_b is the limit $\aleph_o$ of the sequence:

$\{i\} = 1, 2, 3,\ldots$

we may wonder why the difference in the number of marbles in A and in B at t_b is not the limit $\aleph_o$ of the sequence:

$$\{H^{1+a_i} + b_i H - a_i\} = H, 2H, 3H,\ldots H^2 - 1, H^2 + H - 1, H^2 + 2H - 1, H^2 + 3H - 1,\ldots H^3 - 2\ldots$$

How is it possible for that difference to be 0 t_b? Recall that at t_b both boxes contain the same number of marbles. Notice also that we are discussing the limits of strictly increasing ω-ordered sequences, not on the properties that only apply to finite sequences. And things can get even worse if we take into account the fact that the difference in the number of marbles inside A and B becomes null just at t_b, at the first instant after all marbles have already been added. In effect, we have:

$$\forall t \in (t_1, t_b): \exists t_v \in \{t_i\}: t_v \leq t < t_{v+1}$$

Consequently, at instant t the difference d(t) in the number of marbles inside A and B is

$$d(t) = d_v = H^{1+\text{int}(v/H)} + (v \bmod H)\, H - \text{int}(v/H)$$

which, obviously increases with v and then with t within (t_1, t_b). How can it finally be $d(t_b) = 0$? Is it unreasonable to suspect that there is something amiss here?

In order to begin our second variant of Ross' supertask $\{R_i, t_i\}$ (which hardly differs from the original version), consider an ω-ordered collection of identical marbles $\{m_i\}$ labelled with the successive natural numbers, and assume that at each instant t_i of $\{t_i\}$ we add a group of H marbles labelled from $(i - 1)H + 1$ to iH to an initially empty box A. In addition, the box A is provided with a mechanism M such that it removes the marble with the least index from the box; at the same time a new group of H marbles is added to the box, including the first group. The mechanism M is set in such a way that it only works within the interval [ta, tb). In these conditions each marble m_i will be removed at the instant t_i, an instant at which the box will contain exactly $i(H - 1)$ marbles. Thus, as the supertask $\{R_i, t_i\}$ progresses, the number of marbles within A varies according to the following strictly increasing and ω-ordered sequence of natural numbers:

$$\{i(H - 1)\} = 1(H - 1), 2(H - 1), 3(H - 1), 4(H - 1),\ldots$$

On the one hand, and each marble m_i being removed from the box at the precise instant t_i, the one-to-one correspondence $f(t_i) = m_i$ proves that at t_b, once the supertask $\{R_i, t_i\}$ is completed, all marbles have been removed from the box. There is nothing to discuss here. The conclusion that at t_b the box A is empty is a direct consequence of a simple bijection.

On the other hand, let t be any instant within (t_1, t_b), being t_b the limit of the sequence $\{t_i\}$ we will have:

$$\forall t \in (t_1, t_b): \exists t_v \in \{t_i\}: t_v \leq t < t_{v+1}$$

Consequently, at instant t the number n(t) of marbles within the box A will be:

$$n(t) = n(t_v) = v(H - 1)$$

which strictly increases with v, and then with t within (t_1, t_b). It is then impossible for box A to be empty at any instant within the interval (t_1, t_b). Therefore, and taking into account that at t_b no marble has been removed from the box (because at t_b the supertask $\{R_i, t_i\}$ has already finished and the mechanism M is off), the box cannot be empty at t_b. There is nothing to discuss here either. The conclusion to this variant of Ross' paradox can only be that box A at t_b is and is not empty.

To end this section on marbles and boxes, consider the same-labeled ω-ordered collection of marbles $\{m_i\}$ of the above supertask $\{R_i, t_i\}$, and in the place of box A, a hollow cylinder C of an infinite length and a diameter equal to the diameter of the marbles. Assume that at each instant t_i of $\{t_i\}$ the marble m_i is introduced into the cylinder through its left end. At t_b all marbles will have been introduced into C. If we now introduce a rigid rod through the left end of C, the rod may hit a marble m_v, proving that only a finite number v of marbles have been introduced into C. But it may also be the case that the rod traverses the whole cylinder without hitting any marble, as there is no last marble to be hit in the ω-ordered sequence of marbles $\{m_i\}$, which contradicts the fact that infinitely many marbles were introduced inside the cylinder.

Synchronising a supertask

To illustrate the need for H0 in supertasks discussions, a mathematical supertask will be now synchronically performed with a classical supertask. In this case, the classical supertask will be carried out with the collaboration of the infinitely patient guests of Hilbert's Hotel. But before beginning, let us relate some of the extraordinary properties of this illustrious hotel. Its director, for example, has just discovered a new infinitist way of getting rich: he demands one euro from G_1 (the guest of room R_1); G_1 recovers his euro by demanding one euro from G_2 (the guest of room R_2); G_2 recovers his euro by demanding one euro from G_3 (the guest of room R_3); and so forth. Finally, each guest recovers his euro since there is not 'last guest' losing his money. The crafty director then demands a second euro from G_1 who recovers it again by demanding one euro from G_2, who recovers it by demanding one euro from G_3, etc. This way, there are thousands of euros coming from the (infinitist) nothingness to the pocket of the fortunate director! We could also imagine a perpetuum mobile functioning by powering an appropriate machine fed with the calories obtained from the successive rooms of the prodigious hotel in the same way its director got his money

Eccentricities aside, let us assume that the rooms of the inclitus hotel are disposed in a unique row divided into two adjacent parts, the left and the right side. The right side is an ω-ordered sequence of contiguous rooms labelled from left to right as R_1, R_2, R_3,…The left side is also an ω-ordered sequence of contiguous rooms, now labelled from right to left as …L_3, L_2, L_1 and in such a way that L_1 is contiguous to R_1. Symbolically:

$$HH = \ldots L_5\ L_4\ L_3\ L_2\ L_1\ R_1\ R_2\ R_3\ R_4\ R_5\ldots$$

In addition to its front entrance door, each HH's room has two lateral doors, the left door that communicates with the contiguous room on the left, and the right door that communicates with the contiguous room on the right. We will also assume that along the interval $[t_a, t_b]$ all lateral doors are open, and that all entrance doors of every single room are blocked, so that no guest can leave the hotel. To denote that a room L_i or R_i is empty we will write L_i^o, R_i^o; to denote that the guest G_n occupies them, we will write L_i^{Gn}, R_i^{Gn}. We will assume that, initially, each right room R_i, is occupied by the guest G_i, being all left rooms initially empty. So the initial state of HH at t_a will be:

$$HH(t_a) = \ldots L_5^o\ L_4^o\ L_3^o\ L_2^o\ L_1^o\ R_1^{G1}\ R_2^{G2}\ R_3^{G3}\ R_4^{G4}\ R_5^{G5}\ldots$$

Now consider the following HH-change: through the left door of his room, the guest G_1 changes to the left empty room contiguous to his current room, provided that such an empty room exist, and each guest $G_{i,\, i>1}$ changes through the left door of its current room to the room previously occupied by G_{i-1}:

$HH(t_1) = \ldots L_5^{o}\ L_4^{o}\ L_3^{o}\ L_2^{o}\ L_1^{G1}\ R_1^{G2}\ R_2^{G3}\ R_3^{G4}\ R_4^{G5}\ R_5^{G6}\ldots.$

$HH(t_2) = \ldots L_5^{o}\ L_4^{o}\ L_3^{o}\ L_2^{G1}\ L_1^{G2}\ R_1^{G3}\ R_2^{G4}\ R_3^{G5}\ R_4^{G6}\ R_5^{G7}\ldots.$

$HH(t_3) = \ldots L_5^{o}\ L_4^{o}\ L_3^{G1}\ L_2^{G2}\ L_1^{G3}\ R_1^{G4}\ R_2^{G5}\ R_3^{G6}\ R_4^{G7}\ R_5^{G8}\ldots.$

…

By induction or by Modus Tollens (MT) it can be easily proved that for every natural number v it is possible to carry out the first v HH-changes (Theorem 1).

On the other hand, let $A_0 = \{a_1, a_2, a, \ldots\}$ be an ω-ordered set, and consider the following ω-ordered sequence of recursive definitions $\{D_i(A_i)\}$ of the sequence of nested sets $\{A_i\}$:

$i = 1, 2, 3, \ldots \quad D_i(A_i)\text{:}\ \ A_i = A_{i-1} - \{a_i\}$

Let us assume that at each instant t_i of the sequence of instants $\{t_i\}$ the ith definition D_i of $\{D_i(A_i)\}$ is carried out, and that at the same instant t_i the ith HH-change is also carried out, provided that it can be performed. At t_b, once the infinitely many successive definitions D_i has been performed, (supertask $\{D_i, t_i\}$) and thanks to H0, we will have a new sequence of nested sets $\{A_i\}$ completely defined as a complete totality, and we can make any subsequent use of it, for instance to prove a certain theorem. This is standard infinitist mathematics (if it was not for the fact that standard infinitist mathematics is not interested in timetabling the steps of ω-ordered sequences of steps).

Things are quite different for $\{HH_i, t_i\}$. In fact, at t_b, and once all possible HH-changes have been performed, all guests mysteriously disappeared from the hotel: G_n being any guest, she cannot be in any right room R_k (for any natural number k) nor in any left room L_p (for any natural number p) because in the first case only the first n - k HH-changes would have been carried out, while in the second case that number would be p + k - 1. Clearly, both results contradict Theorem 1. So, if H0 applies to supertask $\{HH_i, t_i\}$ in the same way it applies to the recursive definition $\{D_i(A_i)\}$, we have a serious conflict with the ω-ordering derived from the Axiom of Infinity. If not, a convincing reason should be giving to explain why we are not facing a conflict.

Mathematical supertasks

Definitions, recursive definitions, procedures and proofs consisting of infinitely many successive steps are quite common in contemporary mathematics. In general, mathematicians are not interested in the way those infinitely many steps could be actually carried out. They simply take it for granted they are, and focus their attention on the final results. If these results consist of infinite collections, like sets or sequences, they are considered as completed totalities accordingly to the hypothesis of the actual infinity subsumed into the Axiom of Infinity, which in turn implies that the infinitely many steps have been completed in an effective way. Of course, all those infinite definitions, procedures or proofs assume H0. Otherwise, after performing the infinitely many steps of the corresponding definitions, procedures or proofs, we would find ourselves in the odd situation of being incapable of saying anything about them, becoming absolutely useless for us. Uninteresting as it may seem from a pure mathematical perspective, we could schedule those infinitely many successive steps, for instance, in the form of a supertask. These mathematical supertasks have the advantage of not being affected by the problem of change. On the other hand, they have the same discursive functionality as standard supertasks, and could be used, for example, to examine some transfinite fundaments like ω-order and the Axiom of Infinity. Though not in a detailed form (detailed arguments in [12]), we will now introduce some of these mathematical supertasks.

<u>Lost in exchanges</u>

Let $\{a_i\} = a_1, a_2, a_3, \ldots$ be an ω-ordered sequence, and consider it as a table of one row and infinitely many columns ω-ordered and indexed by the successive natural numbers. Assume now that we successively exchange a_1 with the element placed in the next column on the right of a_1. Let us denote these exchanges as a_1-exchanges. After the first n successive a_1-exchanges we would have:

$\mathbf{a_1}, a_2, a_3, a_4, \ldots a_n, a_{n+1}\ a_{n+2}, a_{n+3} \ldots$

$a_2, \mathbf{a_1}, a_3, a_4, \ldots a_n, a_{n+1}\ a_{n+2}, a_{n+3} \ldots$

$a_2, a_3, \mathbf{a_1}, a_4, \ldots a_n, a_{n+1}\ a_{n+2}, a_{n+3} \ldots$

…

$a_2, a_3, a_4, a_5, \ldots a_n, a_{n+1}\ \mathbf{a_1}, a_{n+2}, a_{n+3} \ldots$

It is easy to prove by induction or by MT that for any natural number v greater than 0 it is possible to perform the first v successive a_1-exchanges (Theorem 2). Now consider the following supertask $\{a_i, t_i\}$: at each successive instant t_i of $\{t_i\}$ exchange a_1 with the element in the next adjacent column on the right of a_1, provided that such a column does exist, otherwise, stop the supertask. In any case, at t_b the supertask will have finished. Let v be any natural number and assume that at t_b the element a_1 is in the vth column of $\{a_i\}$. If that were the case, the first v a_1-exchanges would not have been carried out, which goes against Theorem 2. Therefore, and being v any natural number, we must conclude that at t_b the element a_1 is no longer an element of $\{a_i\}$. At t_b, a_1 has disappeared from the table in spite of the fact that no a_1-exchange made it disappear.

<u>The next rational</u>

The set Q^+ of positive rational numbers in their natural order of precedence is densely ordered: between any two rationals infinitely many other rationals exist. But Q^+ is also denumerable, and therefore it can be put into a one-to-one correspondence f with the set N of natural numbers. This correspondence f induces a ω-order in Q^+: $\{q_1, q_2, q_3, \ldots\}$, where q_n is the rational number f(n). Therefore, the set of positive rational numbers can be densely ordered (between any two rationals infinitely many other rationals exist) and ω-ordered (between any two successive rationals no other rational exists). This sort of numerical schizophrenia allows us to develop the following argument. Let x be a rational variable whose initial value is 1, and consider the following sequence $\{D_i(x)\}$ of x redefinitions:

$i = 1, 2, 3,..$:
If $|q_{i+1} - q_1| < x$ Then $D_i(x)$: $x = |q_{i+1} - q_1|$
Else $D_i(x)$: x remains unchanged

which redefines x (for each i = 1, 2, 3 ,…) as $|q_{i+1} - q_1|$ if $|q_{i+1} - q_1|$ is less than the current value of x, $|q_{i+1} - q_1|$ being the absolute value of $q_{i+1} - q_1$, and < the natural order in Q^+.

It is worth noting that all of the successive definitions $D_i(x)$ redefine the same object, the rational variable x. By contrast, each recursive definition $D_i(A_i)$ of the above sequence of definitions $\{D_i(A_i)\}$ defines a different object, the set A_i of the sequence of nested sets $\{A_i\}$. This difference is crucial: in the second case we have a ω-ordered sequence of definitions without a last definition that originates a ω-ordered sequence of nested sets without a last set. In the second case, we also have a ω-ordered sequence of definitions without a last definition, but all those successive definitions define the same object x, which forces $\{D_i(x)\}$ to leave a permanent trace in the form of the rationality of x: once all possible redefinitions $D_i(x)$ are performed, the variable x will continue to be a rational variable, and one that has been redefined a certain number of times. Otherwise, H0 would have been violated, and the same violation could happen to any sequence of successive definitions of the same object or of different objects. The only alternative to this, would be the prohibition to redefine the same object infinitely many successive times, in which case that prohibition had to be declared as an additional axiomatic restriction in transfinite mathematics. Meanwhile, we continue our argument on $D_i(x)$ since, for the time being, we can redefine an object any finite or infinite number of successive times.

By induction (or by MT) it can easily be proved that for any natural number v, the first v redefinitions of the sequence $\{D_i(x)\}$ can be carried out (Theorem 3). Consider the following supertask $\{D_i(x), t_i\}$: perform $D_i(x)$ at instant t_i if it is possible to perform $D_i(x)$; if not, end the supertask. In any case, at t_b all possible redefinitions $D_i(x)$ will have been carried out. Whatever the value of x at t_b is, it will be a rational number since x is a rational variable, one that can only take rational values, and that has been redefined a certain number of times. Otherwise, we would be violating H0. We now prove the following two contradictory results on the value of x at t_b.

R1: At t_b the rational $q_1 + x$ is not the least rational greater than q_1.

Proof: Q Being densely ordered, the rational number $q_1 + 0.1x$, for instance, is greater than q_1 and less than $q_1 + x$. So $q_1 + x$ is not the least rational greater than q_1.

R2: At t_b the rational $q_1 + x$ is the least rational greater than q_1.

Proof: Assume it is not. Q^+ being ω-ordered, there will be a q_v in $\{q_1, q_2, q_3,\ldots\}$ such that:

$$q_1 < q_v < q_1 + x$$

And then:

$$0 < q_v - q_1 < x$$

which implies the vth redefinition $D_v(x)$ (that would have defined x as $q_v - q_1$) has not been carried out, a conclusion that is impossible according to Theorem 3. So $q_1 + x$ is the least rational greater than q_1.

Cantor's 1874 argument

In 1874 Cantor published a short paper in which he proved that the set A of algebraic numbers (and then the set Q of rational numbers) is denumerable [10]. In the same paper he also proved, for the first time, the set R of real numbers is not denumerable. This 1874 argument is far less well known than his diagonal proof, yet it is as conclusive as the other. Cantor's 1874 argument leads to three exhaustive and mutually exclusive alternatives, each of them proving that R is not denumerable. But two of these three alternatives could also be applied to the set Q of rational numbers. Thus, we need to prove that Cantor's 1874 argument always leads to the third alternative when it is applied to the set of rational numbers, otherwise Q would also be non-denumerable. So, until this is proved to be the case (and the proof is far from being obvious) set theory is facing a contradiction related to the cardinality of the set of rational numbers. It is hard to believe that neither Cantor nor his infinitist successors ever realized that proof is in fact necessary.

A variant of Cantor's 1874 argument will serve us to define a new mathematical supertask with conflicting consequences. As noted above, Q can be put into a one-to-one correspondence f with N. We can therefore define an ω-ordered sequence of rational numbers $\{q_i\} = \{f(i)\}$ that contains all rational numbers. Let (a, b) be any rational interval and x a rational variable whose domain is just (a, b) and whose initial value is c, any element of (a, b). Then consider the following ω-ordered sequence $\{D_i(x)\}$ of successive x redefinitions:

> i = 1, 2, 3,…:
> If $q_i \in (a, b)$ And $q_i < x$ Then $D_i(x)$: $x = q_i$
> Else $D_i(x)$: x remains unchanged

that compares x with the successive q_i of $\{q_i\}$ within (a, b), and redefines x as q_i each time q_i is in (a, b) and is less than the current value of x. By induction or by MT it is easy to prove that for each natural number v it is possible to perform the first v definitions of the sequence $\{D_i(x)\}$ (Theorem 4).

Assume that if performing $D_i\{x\}$ is possible then it is performed at the precise instant t_i of the sequence $\{t_i\}$, otherwise end the supertask $\{D_i(x), t_i\}$. In any case, at t_b all possible redefinitions of the sequence $\{D_i(x)\}$ will have been carried out. According to H0, at t_b x will be defined as a rational number within the interval (a, b), since x is a rational variable whose domain is (a, b) and has been redefined a certain number of times. Consider then the rational interval (a, x), and let q be any of its elements. Evidently q is in (a, b) because (a, x) is a subinterval of (a, b). Yet q cannot be an element of $\{q_i\}$. Let us suppose it is. In this case, we would have that $q = q_v$, for a certain q_v in $\{q_i\}$, and then $q_v < x$ because q_v is in (a, x). But this implies that the vth redefinition $D_v(x)$ has not been carried out and this conclusion contradicts Theorem 4 (note that $D_v\{x\}$ would have been redefined x as q_v). We must conclude that the sequence $\{q_i\}$ that contains all rational numbers does not contain all rational numbers.

Cantor's diagonal argument

Cantor's diagonal argument [11] is one of the most celebrated and productive arguments in the recent history of logic and mathematics. It is a simple and elegant Modus Tollens (MT) proving the set of real numbers is non-denumerable. In our opinion, and despite its critiques, it is correct. Indeed, it is relatively common in infinitist discussions to try to discard an argument because the conclusion of another independent argument contradicts the conclusion of the first one. This has been the case with Cantor's diagonal argument, and it is clearly inadmissible, because if two independent arguments prove contradictory results they are simply proving a contradiction. An argument can only be dismissed if we indicate where and why that particular argument fails.

Cantor's diagonal argument also poses a problem that has not been adequately addressed so far: could the indexed rows of Cantor's table be permuted in such a way that the resulting table defines a rational diagonal (and then a rational antidiagonal)? Clearly, and for the same reasons as in Cantor's 1874 argument, if that were the case, we would be facing a contradiction regarding the cardinality of the set of rational numbers. As with the alternatives of Cantor's 1874 argument, we need to prove that such a reordering of the rows of Cantor's table is not possible if we want to discard the contradiction (and, again, the proof is anything but obvious). Once again, it is amazing how little attention has been paid to this problem.

Our last mathematical supertask is related to Cantor's diagonal. Although in this case some auxiliary work will be necessary. To begin with, we need to prove the following theorem of the nth decimal:

> For every natural number n there are infinitely many different rationals in (0, 1) with the same decimal d_n in the same nth position of its decimal expansion.

Without going into details, the sequence of rationals numbers:

$q_1 = 0.d_1d_2 \ldots d_n1$

$q_2 = 0.d_1d_2 \ldots d_n11$

$q_3 = 0.d_1d_2 \ldots d_n111$

$q_4 = 0.d_1d_2 \ldots d_n1111$

…

and the bijection $f(n) = q_n$ will suffice to prove the theorem. Let us recall that Cantor's hypothetical indexed table $\{r_i\}$ contains all real numbers (rational and irrational) within the interval (0, 1), one number in each row r_i. Although it is not necessary for the next argument, Cantor's table could easily be redefined in order to ensure it contains at least all rational numbers within (0, 1).

The decimal expansion of rational numbers with a finite decimal expansion will be completed with infinitely many 0s on the right of its last decimal. So, in the place of 0.25 we will write 0.250000… Finally, we will say that a row of Cantor's table is n-modular if the nth decimal of its decimal expansion is (n mod 10). For instance:

row r_1: 0.60**3**0**5**111022**3**39… is 3-modular, 5-modular, 13 modular etc.

row r_2: 0.0**2**000**67**1010000… is 2-modular, 6-modular. 7 modular etc.

row r_3: 0.**1**1**3**00000000000… is 1-modular, 3-modular. 10 modular etc.

If the nth row r_n of Cantor table is n-modular, we will say that it is D-modular (in the above examples the rows r_2 and r_3 are D-modular). If a row r_i is not D-modular it can be exchanged with any of the following i-modular rows $r_{j,\, j>I}$, provided that such a row exists. Once exchanged, r_i will contain the number in r_j (and vice versa) and it will become D-modular. We call D-exchanges to these exchanges. This is all we need to define the next Cantor diagonal supertask.

Assume that at each instant t_i of $\{t_i\}$ the row r_i of Cantor's table is considered in such a way that:

If r_i is D-modular then it remains unchanged.

If r_i is not D-modular and it can be D-exchanged with any following i-modular row $r_{j, j>i}$, then it is D-exchanged.

If r_i is not D-modular and cannot be D-exchanged then it remains unchanged.

Note that once a non-D-modular row r_i has been D-exchanged, it becomes D-modular and will remain D-modular (and unaffected by the subsequent D-exchanges) due to the condition $j > i$ (in $r_{j, j>i}$) of D-exchanges. At t_b all rows will have been considered and the supertask will have finished. Let this supertask be denoted by $\{r_i, t_i\}$.

It is now immediate to prove that at t_b, once performed $\{r_i, t_i\}$, all rows of Cantor table are D-modular. Let us assume that they are not, i.e let us assume that at t_b there is a row r_n that is not D-modular. As a consequence of ω-asymmetry, r_n has a finite number (n-1) of preceding rows and an infinite number of succeeding rows. This implies that r_n can only be preceded by a finite number of n-modular rows. According to the theorem of the nth decimal, there are infinitely many rationals with the same decimal (n mod 10) in the same nth position, since all n-modular rows have the same decimal (n mod 10) in the same nth position of its decimal expansion. Or in other words, there are infinitely many n-modular rows, of which only a finite number precede r_n. Consequently, r_n is succeeded by infinitely many n-modular rows and hence it had to be D-exchanged with any of them. Therefore r_n must be D-modular. We must conclude that, once the supertask $\{r_i, t_i\}$ has been performed, all rows of the Cantor table are D-modular. As in the case of Cantor's diagonal argument, this one is also a simple Modus Tollens. The reader can easily prove that supertask $\{r_i, t_i\}$ leads to other conflicting results such as the disappearance of infinitely many rows from the table.

Now then, if all the rows of Cantor table become D-modular, then the new diagonal of the table will be a periodic rational number within (0, 1) whose period is 1234567890, i.e. the rational number:

0.123456789012345678901234567890…

From this diagonal we can define infinitely many rational antidiagonals, for example, the periodic rationals within (0, 1) of periods 0123456789, 45, 21, etc. For the same reasons as with irrational antidiagonals, each of these rational antidiagonals would prove that the set of rationals within (0, 1) is not denumerable. Therefore, we would have a fundamental contradiction in set theory: the set of rational numbers would and would not be denumerable. The alternative to this contradictory conclusion would be a violation of H0, so that, for example, at t_b the rows of Cantor's table are no longer real number within (0, 1), or its rows are arbitrarily permuted so that we cannot ensure that all of its rows are D-modular, etc. Obviously, the same final arbitrary effects could be expected in any definition, procedure or proof composed of infinitely many successive steps, and transfinite mathematics would lose all of its meaning.

Partitions a la Cantor

The following summarized argument is not a supertask but a mathematical procedure of infinitely many steps, expressed in the compact form of computer language, that illustrates another way of posing problems related to the hypothesis of the actual infinity. It is inspired by Cantor Ternary Set (Cantor's power) and by Cantor's 1885 argument [12] on the partition of the real line. Let A be the real interval (0, 1) and X a set of indexes whose elements will be referred to as a, b, c, d,… and whose cardinal is $2^{\aleph_o}$. Let u and v be two real variables and f a one-to-one correspondence between A and X. Consider the following procedure P:

```
u = 0                   'Used to define the left ends of the successive intervals.
v = 0                   'Used to define the right ends of the successive intervals.
While A ≠ ∅
   X = X – {a}          'a (b, c, d,…) is any X's element.
   A = A – {f(a)}       'Remove f(a) from A.
   v = v + f(a)         'Right end of the interval.
   If v is in R Then    'R is the set of proper real numbers.
      (xa, ya] = (u, v] 'Define a new real interval.
   Else
      Exit Loop         'Two real numbers whose sum is not a proper real number.
   End If
   u = v                'Left end of the next adjacent and disjoint interval.
Loop
```

Since the sum of any two real numbers is a real number, P exhausts the sets I and A, and defines a partition $T = \{(x_a, y_a], (x_b, y_b], (x_c, y_c],\dots\}$ in the real line, each of whose intervals $(x_h, y_h]$ defines a different real number $r_h = y_h - x_h$. It is then immediate to prove that $g((x_h, y_h]) = r_h$ is a one to one correspondence between T and (0, 1). Thus, by selecting a rational number q_h within each $(x_h, y_h]$ we would have a non-denumerable sequence of different rational numbers. As in the cases of Cantor's 1874 argument and Cantor's diagonal presented above, this new conclusion also points to a possible contradiction related to the cardinality of the set of rational numbers.

Infinity and Physics

While many authors believe in the formal consistency of supertasks, the number of them who also believe in the physical possibilities of actually performing a supertask is much smaller. In this latter group we also find those who believe that supertasks could be performed in infinite intervals of time perceived as finite intervals thanks to the relativistic dilation of time (bifurcated supertasks). As we will see now, ω-asymmetry and quantum mechanics add new difficulties to the possibility of actually executing (or observing) a supertask in a finite interval of time. In fact, let t_p be Planck time (5.39×10^{-44} s) and consider the time interval $(t_b\text{-}t_p, t_b)$. Due to ω-asymmetry, at any instant t within $(t_b\text{-}t_p, t_b)$ only a finite number of actions have been performed, and there still remains an infinite number of actions to be performed. Indeed, since t_b is the limit of $\{t_i\}$, there will exist a t_v in $\{t_i\}$ such that $t_v \leq t < t_{v+1}$, and hence, at instant t only a finite number v of actions will have been carried out. So, infinitely many actions would have to be performed within an interval of time far less than Planck time.

A simple exercise of differential calculus proves that, assuming Heisenberg Principle of Uncertainty, Planck length and Planck time are the shortest length and least time that can be measured in physical terms. So, we could never verify in physical (proper or improper) terms that a supertask has been carried out in a finite interval of time simply because infinitely many actions would have to be carried out in less than Planck time. Furthermore, as most of physicists now suspect, beyond Planck scale physical laws may no longer hold. It is also quite plausible that nothing in the physical world can last a time less than Planck time. This adds some additional difficulties to the pretension that supertasks could be physically carried out. In any case, here we have only focused on conceptual ω-supertasks. Our aim was not to implicate physics in supertask theory but to illustrate the way supertasks could be used to call into question the Axiom of Infinity because, as we will see now, that questioning is of great interest to experimental sciences e.g., physics.

Apart from ω-asymmetry, ω-inconsistency also applies to physical supertasks: regarding the number of actions to be performed, that number can only take two values: $\aleph_0$ and 0. The only solution to this dichotomy is that infinitely many successive actions have to be carried out simultaneously. But successive actions cannot be performed simultaneously in physical terms, whether proper or improper.

The mathematical infinite, on the other hand, is anything but a trivial matter. Its consequences in physics are enormous. The points of the infinitist continuum of the real numbers, for instance, are of capital importance in physics: point masses, point particles, point charges etc. The special theory of relativity, one of the more successful theories of modern physics, is a physical theory on the spacetime continuum. Theoretical physics is made almost exclusively of infinitist mathematics. But, as we will see, the continuum may not be the best model to deal with the physical world, particularly when we approach certain ultramicroscopic scales, such as the Planck scale. The persistent problems we have found there (in the physical realm) for more than fifty years suggest the physical world could be essentially discrete, discontinuous, i.e., digital.

For many contemporary physicists, the persistent incompatibility between quantum mechanics and general relativity is a consequence of the lack of discreteness of the continuum-based models [13]. They suspect that space and time are not continuous but discrete (composed of indivisible minima), and that the granular fabric of spacetime could be the meeting place for both fundamental theories of physics. An increasing number of theoretical and experimental research is now trying to prove the discrete nature of space and time. As we will also see here, the search for violations of Lorentz symmetry at Planck scale (the plausible granular scale of the physical world) is now becoming an active area of theoretical and experimental research [14], [15]. The continuum being a formal descendent of the Axiom of Infinity, we must insist on the importance of reexamining the formal consistency of the actual infinity hypothesis, simply because if it were inconsistent, so would all continuums. But most physicists, who fully rely on mathematics, have not even thought of that possibility. And, for obvious reasons, the infinitist paradise is not very enthusiastic about it.

Some of the above conclusions on supertasks and mathematical supertasks suggest that the hypothesis of the actual infinity (the existence of infinite totalities as complete totalities) embraced by the Axiom of Infinity may be inconsistent. If that were the case, more than a century of mathematics would have to be revised. And perhaps a new type of discrete (digital) mathematics would have to be developed, including discrete analysis and discrete geometry (see below). Physics would also be affected by this hypothetical crisis of infinity, though in a different way. At least this is what the following points seem to be indicating:

1. Though theoretical physics is made up of infinitist mathematics, experimental physics always deals with digital results. Even when we call them analog, all observations and measurements are always discrete, truncated to a small number of digits.
2. When the infinites appear in physics equations they have to be removed from them in order to avoid the unsolvable problems they invariably lead to, for example, in the Standard Model of Particles (renormalization).
3. All physical magnitudes seem to be of a discrete nature, with indivisible minima. Even space and time are also suspected of being also composed of indivisible minima.
4. Nothing we have observed, measured or divided is infinite. In physics, in fact, infinity could be just a ‘manner of speaking’.
5. The suspected digital scale of nature could be the Planck scale. When experimental and theoretical physics approach this scale some problems appear, and most physicists suspect that it is not possible to fin a solution within our current analog models. We will deal with one of these problems in this section.

If nature is indeed discrete in all its observables, the analog mathematics of the continuum may not be the most appropriate instrument to deal with the physical world. While we examine the world far away from its discrete scale of minima, the analog mathematics of the continuum works quite well. The problems appear when we approach that scale.

Before approaching to the hypothetical digital scale of the physical world (and the problems it poses to some well-established physical theories) let us recall n-expofactorial numbers of unimaginable size as $9^{!9}$. It is one thing to be able to define finite numbers that large, but to suppose that they are meaningful is quite another. Imagine a real number with $9^{!9}$ decimals in its decimal expansion. Imagine as well that all physical constants needed to explain our Universe were real numbers with $9^{!9}$ decimals (and $9^{!9}$ is minuscule when compared with $\aleph_o$). In our opinion, that would be a really grotesque universe (we presume Ockham would also agree with us). All periodic rational numbers and all irrational numbers, on the other hand, have a ω-ordered sequence of $\aleph_o$ decimals in their decimal expansions. Also, according to the infinitist orthodoxy, all of them exist all at once as a complete and finished totality. Most of the irrational numbers within (0, 1) are supposed to have an infinite (ω-ordered) and random sequence of decimals. Being infinite and random as they are, it would be easy to prove that each of these sequences contains an encoded version of all known texts (books, papers, poems, letters, etc.) written by all humans from Neolithic to nowadays. And what is even more incredible, each one of them would also contain sequences of the same decimal repeated, for instance, $99^{!99}$ times. These are inevitable consequences of the marriage between randomness and the actual infinity.

As we know, $\aleph_o$ is the smallest number greater than all finite natural numbers, the cardinal of the set of natural numbers. The problem with $\aleph_o$ is that its definition is not related to the operational definition of natural numbers via the successor set. $\aleph_o$ is not the successor of any finite natural number simply because there is no last natural number to be succeeded by $\aleph_o$. Cantor proved $\aleph_o$ is not a natural number because $\aleph_o = \aleph_o + 1$, while every natural number n satisfies $n \neq n + 1$. Then he proved that $\aleph_o$ is greater than all finite cardinals because for any finite cardinal n the set {1, 2,… n} is not equivalent to N, while it is a proper part of N. He also proved that it is the least cardinal greater than all finite cardinals ([4], theorems A and B, epigraph 6). So, $\aleph_o$ is the limit of all strictly increasing and ω-ordered sequence of natural numbers. But $\aleph_o$ does not play in physics the fundamental role it plays in set theory.

The next transfinite cardinals are $2^{\aleph_o}$ y $\aleph_1$. The former is the cardinal, among others, of the set of real numbers, i.e., the power of the continuum. The latter is the cardinal of the set of all ordinals whose sets have the same cardinal $\aleph_o$. We do not know if $2^{\aleph_o} = \aleph_1$ (hypothesis of the continuum). The sequence of powers ($2^{\aleph_o}$, $2^\wedge 2^{\aleph_o}$, $2^\wedge 2^\wedge 2^{\aleph_o}$,…) and the sequence of alephs ($\aleph_1$, $\aleph_2$, $\aleph_3$,…) yield an interminably number of infinities of an increasing infinitude that leads towards the absolute infinity, which is so infinite that it becomes inconsistent, at least to our poor limited human minds, says Cantor. All of them, except the power of the continuum, are absolutely irrelevant to physics.

The spacetime continuum is, in fact, grounded on the continuum of the real numbers, whose cardinal is $2^{\aleph_o}$ (in physics literature, even under the signature of certain Nobel laureates, it is not unusual to meet with the erroneous assertion that this cardinal is $\aleph_1$. After all, transfinite arithmetic may not be essential for doing physics). As is well known, the continuum of the real numbers is so infinite that a straight line segment of Planck length (1.62×10^{-35} m) has the same number of points as the whole tridimensional visible universe, which, bijections aside, is rather enigmatic from a pure physical point of view (think, for instance, of virtual quantum particles). In our opinion, bijections and ellipsis form a really dangerous couple.

The physical theory more directly concerned with the hypothesis of the actual infinity is the special theory of relativity, since it was founded on the basis that space and time are unified into a four-dimensional continuum called spacetime, this continuum being the infinitist continuum of real numbers. As is well known, Einstein’s theory refines Newton mechanics for velocities approaching the speed of light. As in the case of Newton mechanics, the special theory of relativity has been satisfactorily confirmed by experiments and observations. But a question becomes inevitable: is there any aspect or scale of nature at which the special theory of relativity will also need to be refined, or is it the ultimate theory? The question is pertinent because space and time could be discrete rather than continuous, in whose case some aspects of the theory of relativity would need to be modified.

Since the beginning of the 21st century there is a growing interest in the search for violations of Lorentz symmetry at the Planck scale. Although this scale was intended to define a metric reference independent of our arbitrary definitions of unities for mass, length and time, the interest in the Planck’s scale has gone far beyond the original metric objectives of its author. Indeed, now it is considered as an appropriate candidate to define the granularity (discreteness) of space and time. The Planck scale is defined by a set of universal constants, basically Planck mass m_p, Planck length l_p and Planck time and t_p (Planck energy, Planck charge and Planck temperature can also be included). Planck mass, Planck length and Planck time, are defined in terms of three universal constants: h (Planck constant), c (the speed of light in the vacuum) and G (constant of gravitation). The universality of the Planck universal constants poses, as we will now see, some significant problem to Lorentz transformation.

The First Principle of the special theory of relativity asserts that the laws of physics are universal, the same in all inertial reference frames. The universality of physical laws implies the universality of the physical constants involved in their mathematical expressions. In addition, if A and B are two universal constants, any algebraic combination of them must also be universally constant. For instance, being μ_o (the magnetic permeability of the vacuum) and ε_o (the electric permittivity of the vacuum) two universal constants, the algebraic combination of them $(\mu_o \cdot \varepsilon_o)^{-1/2}$ is also a universal constant (in this example, the speed c of light in the vacuum). For the same reason, l_p, t_p and m_p, which are also defined as algebraic combinations of three universal constants (h, c and G in the three cases), can only be universal constants in all reference frames. If that were not the case because of a certain algebraic combination f(h, c, G) changed with relative motion, this particular change could only be due to a change in, at least, one of these three universal constants (provided that real numbers and algebraic operations do not change with relative motion). Thus, at least one of those three universal constants would change with relative motion and it would not be the universal constant it was assumed to be.

The problem is that Lorentz transformation does not preserve the universality of certain algebraic combinations of h, c, and G. More specifically, it does not preserve the universality of Planck length, Planck time and Planck mass. As L. Smolin pointed out [14], it is really astonishing that this problem (the relativity of universal constants) had never been posed until the first years of the 21st century. It is a good example of the cuasi-religious way we learn, teach and profess science in general, and certain theories in particular. At the beginning of the 21st century, Amelino-Camelia proposed a solution that is now known as Doubly Special Relativity, also known as Deformed Special Relativity (DSR for short) [16]. In addition to the speed of light as universal constant, DSR includes two additional universal constants (independent of relative motion), a maximum energy (Planck energy) and a minimum length (Planck length). The theory has now several variants as DSR II by J. Magueijo and L. Smolin [17]. Not surprisingly, DSRs have not been enthusiastically welcomed.

DSR and its successive variants have built upon the same infinitist mathematics of the continuum (as the rest of physical theories did). The problem here is that at Planck scale we plausibly approach the discrete scale of nature, a place where the continuum-based mathematics could no longer be the appropriate instrument. It is at this point where the importance of examining the formal consistency of the hypothesis of the actual infinity (subsumed into the Axiom of Infinity) becomes evident: if that hypothesis were inconsistent so would all continuums formally derived from it, and we would be forced to develop a new discrete mathematics more attuned to the physical world (the branch of mathematics we usually call discrete mathematics has nothing to do with this issue). Besides, and for the first time in the history of logic and mathematics, we now have at our disposal two productive instruments to put into question that foundational hypothesis: On the one hand ω, the least transfinite ordinal, with its trail of asymmetries, dichotomies, and more than possible inconsistencies; and, on the other hand, supertask theory, an appropriate scenario to represent the arguments.

Platonism and biology

From a physical point of view, an object exists if it can interact with other objects in such a way that their states result somehow modified as a consequence of this interaction. It is just through these interactions that we can detect the existence of physical (spacetemporal) objects. The different branches of physics (and in general of experimental sciences) study the different types of interactions, from a simple change in the trajectory of a photon to a chemical reaction or a galactic collision. We call them 'dynamic interactions' because energy is always involved in them. As far as we know, they are always governed by the same set of universal laws. No exception is known to this consistent behaviour of nature: rivers always flow downhill.

Living beings introduced another type of interaction into the physical world: 'infodynamic interactions', interactions in which arbitrary signals and codes are involved. Infodynamic interactions modify the state of the receiving objects in such a way that it is not always possible to infer the changes from the physical laws but from the complex evolutionary and reproductive history of each organism. Obviously, all objects involved in infodynamic interactions are also physical objects subjected to dynamic interactions with the rest of the world. Apart from arbitrary, infodynamic interactions are also teleonomic, the purpose in most of the cases being directly or indirectly related to reproduction (which also includes surviving), the universal goal of all living beings.

Living beings survive and reproduce in a physical environment that, as noted, is consistently driven by a universal set of physical laws. It is not surprising that living beings behave in concert with that legal consistency in order to survive and reproduce, and that somehow nature consistency had been finally captured in genetic, epigenetic and neurological terms. Or in other words, it should not be surprising that we had finally become aware of the fundamental laws of logic, and capable of developing formal systems in accordance with the physical world.

Perhaps confused by this natural harmony between our formal abilities and the formal consistency of the physical world, we could have magnified the ontological status of mathematical objects (Platonism). But let us recall that the same mathematical objects are also essential to many erroneous physical theories. They are also useless to account for most chemical, geological and biological phenomena, let alone psychological, or sociological ones. Furthermore, if the above arguments on the hypothesis of the actual infinity were really conclusive, and this hypothesis were finally proved to be inconsistent, then Platonism would no longer make sense because the sequence of natural numbers could only be potentially infinite. Or in other terms, natural numbers, the simplest mathematical objects, could only be the result of successive mental recursive operations.

Platonism claims that mathematical objects do exist in an even more profound sense than physical objects. Notwithstanding, the only method for us to test the existence of an object is by means of its dynamic or infodynamic interactions with other objects. Non-spacetemporal objects (e.g. abstract objects) are inaccessible for this objective physical test of existence, and therefore no causal relation can be established with them, which inevitably remind us of Benacerraf's epistemological argument against Platonism. To solve this inconvenience, Platonism makes another claim related to the actual existence of mathematical objects, namely that we can access them by means of a cognitive ability usually called intellectual (or mathematical) intuition. We will now examine this hypothetical ability from the perspective of neuroscience.

To paraphrase Zeki [18], the organization and laws of the brain dictate all human activity, and therefore, there can be no real theory of mathematics unless it is neurobiologically based. The kind of mathematical intuition that Platonism calls on to defend the contact between spacetime and non-spacetime objects may seem incompatible with neuronal processing of information.

We tend to think about vision, hearing, or any perceptual experience, as if our brain worked like a camera, capturing the external reality in a shot. But what the brain does is more complex and counter-intuitive: it reacts to stimuli segregating and processing them in different parts of the brain in order to integrate the information, and to give rise to a unitary conscious experience. It does such a thing without us even noticing it.

The brain, in brief, fabricates all perceptions. As a tireless artisan, it is constantly processing sensory outputs, categorizing them, and making generalizations and abstractions that allow us to represent reality. Plato believed that objects are derived from abstraction, but neuroscience has taught us that it is the other way round. This capacity of abstraction and concept formation is primitive, allowing us to attain knowledge by means of binding information and constructing the object of perception.

The importance of this finding cannot be underestimated, because otherwise there would not be neurons enough to represent all that exists. In other words, if knowing consisted of looking things up in a sort of mental repertoire in which each single object had to be previously registered, the pages we needed for such a catalogue would be endless, whereas our physical memory and space is limited. We may discuss whether or not numbers are infinite but there is no doubt that the number of neurons each of us have is quite finite. Fortunately for us, the process of abstraction saves a lot of energy, neuronally speaking.

For instance, when we look at an orange, our preconscious experience is not single but fragmented. The brain processes separately its colour, size, shape, and smell before giving rise to the conscious experience of "seeing an orange” in which all these features are perceived altogether (except if one suffers from a serious brain injury). This is how “an orange becomes all the oranges that exist in the world”, and this is how abstraction and generalization take place in the neurophysiological realm [19].

Let us bring another example to illustrate why anatomic acquisition of knowledge is so efficient. Imagine a set of, say, fifty alphabetical symbols. With such a small number of symbols we could create many languages, in which we could write and tell a vast number of stories. Now, imagine that every time we wanted to tell a story we had to start from scratch creating different alphabets (different sets of letters), in each language. It would be very costly in terms of time and energy. On the contrary, reusing and combining the same letters seems to be a wiser strategy.

To skip the problem of having a limited memory, evolution selected brains that identify common features shared by same objects, without having to previously register all of them in order to identify a single one. As for intuition -an ability that mathematics care about-, it probably results from the intrinsic activity in the brain, i.e., the default mode network discovered by Raichle and his colleagues [20], which is believed to play a major role in brain function. We certainly know that intuition, whether mathematical, artistic or naturalistic, is part of the brain knowledge acquiring machinery. Therefore, it is subjected to the same organization and rules that are being revealed by an overwhelming amount of experimental studies.

References

[1] J. F. Thomson, «Tasks and SuperTasks,» *Analysis,* vol. 15, pp. 1-13, 1954.

[2] P. Benacerraf, «Tasks, Super-Tasks, and Modern Eleatics,» *Journal of Philosophy,* vol. LIX, pp. 765-784, 1962.

[3] A. W. Moore, The Infinite, New York: Routledge, 2001.

[4] M. Hallet, Cantorian Set Theory and Limitation of Size, Oxford University Press, 1984.

[5] G. Cantor, Contributions to the founding of the theory of transfinite numbers, New York: Dover, 1955.

[6] M. Black, «Achilles and the Tortoise,» *Analysis,* vol. XI, pp. 91-101, 1950 - 51.

[7] R. Taylor, «Mr. Black on Temporal Paradoxes,» *Analysis,* vol. 12, pp. 38-44, 1951 - 52.

[8] J. Watling, «The sum of an infinite series,» *Analysis,* vol. 13, pp. 39-46, 1952 - 53.

[9] P. Clark y S. Read, «Hypertasks,» *Synthese,* vol. 61, pp. 387-390, 1984.

[10] G. Cantor, «Über verschiedene Theoreme asu der Theorie der Punktmengen in einem n-fach ausgedehnten stetigen Raume Gn,» *Acta Mathematica,* vol. 7, pp. 105-124, 1885.

[11] C. A. Pickover, Keys to Infinity, New York: Wiley, 1995.

[12] A. Leon, Infinity at stake, Madrid: Impreso por Bubok Publishing, 2013.

[13] G. Cantor, «Über eine eigenschaft aller reallen algebraishen zahlen,» *Journal für die reine und angewandte Mathematik,* vol. 77, pp. 258-262, 1874.

[14] G. Cantor, «Über Eine elementare frage der mannigfaltigkeitslehre,» de *Jahresberich der Deutschen Mathematiker Vereiningung*, 1891.

[15] S. Majid, «Quantum space time and physical reality,» de *On Space and Time*, S. Majid, Ed., New York, Cambridge University Press, 2008.

[16] L. Smolin, The trouble with physics, London: Allen Lane. Penguin Books, 2007.

[17] T. Maudlin, Quantum non-locality and relativity, Wiley-Blackwell, 2011.

[18] G. Amelino-Camelia, «Testable scenario for relativity with minimum length,» *Physics Letters B,* Vols. %1 de %2510 (1-4), p. 255–263, 2001.

[19] J. Magueijo y L. Smolin, «Generalized Lorentz invariance with an invariant energy scale,» *Phys. Rev. D,* vol. 67, p. 044017, Feb 2003.

[20] S. Zeki, Una visión del cerebro, Barcelona: Ariel, 1995.

[21] F. Mora, Cómo funciona el cerebro, Madrid: Alianza, 2007.

[22] M. E. Raichle, «The Brain's Dark Energy,» *Science,* vol. 319, pp. 1249-1250, 2006.

[23] R. Dedekind, Stetigkeit und irrationale Zahlen, Braunschweig: Vieweg and Sohn, 1872.