the general science journal

On Cube–Pyramid Coincident Configurations

Dragan N. Turanyanin*

Belgrade, Serbia

This essay presents a three-dimensional analogue of the *Theorem of Coincidence* previously established for regular polygons. Here, a cube is centrally inscribed within a square pyramid under strict coincidence conditions. This geometrical setup possesses two equally significant cases: basic C_1 and rotated C_2. Together with the planar blueprint, this spatial counterpart forms a coherent framework suggesting that strict central coincidences constrain dimensions through exact algebraic laws. Accordingly, connection with golden ratio φ and plastic constant ρ surprisingly emerges.

Introduction

The study of coincidence conditions in geometry has a rich history. Recently [1], it was shown that if a regular polygon is inscribed in a triangle under a central coincidence constraint - namely, the coincidence of circumcenters of the two figures (assuming axial symmetry) - then only highly restricted and elegant solutions emerge. In particular, the triangle itself acts as the minimal constraining case, with solvable algebraic relations that reduce to cubic forms. This result emphasizes both the rarity and the structural beauty of coincidence phenomena in planar geometry.

The present essay develops a three-dimensional analogue of this principle, focusing on the embedding of a cube in a square pyramid under strict coincidence conditions. We shall refer to this as the "Centrally Inscribed Cube" (CIC) Pyramid. The coincidence condition is stated as follows:

1. One cube face lies centrally on the pyramid's square base.
2. The centers of the circumscribed spheres of the cube and the pyramid coincide.

From this setup, a natural theorem emerges — the CIC Pyramid Theorem — which describes the precise algebraic relations between the pyramid and cube dimensions. The Theorem stands as a companion to the earlier planar Theorem of Coincidence. Together, they suggest the beginnings of a general framework: Coincidence Principles in geometry, where strict central alignments between inscribed and bounding figures and solids lead to rare but exact algebraic constraints.

The manuscript is structured as follows: Section with CIC Pyramid Theorem in precise terms. Two equally significant Sections: Configuration 1 and Configuration 2 with similar inner structure (the geometric construction, the algebraic framework and numerical examples, highlighted connections and extensions) and final Section with a discussion of significance and open directions. Appendixes present: A, few figures more and B brings related algebra in more details. Afterwards and finally, Addendum briefly compares the main coincident setup with two non-coincident analogs.

On the Theorem

The following result may possibly be viewed as the three-dimensional extension of the Theorem of

* Independent Researcher; e-mail: turanyanin@yahoo.com

Coincidence established in planar geometry.

CIC Pyramid Theorem

Let a cube be centrally inscribed in a square pyramid under the following conditions:

1. One cube face lies centrally on the pyramid's square base.
2. The centers of the circumscribed spheres of the cube and the pyramid coincide.
3. All eight cube's vertices touche the pyramid's sides.

Then the pyramid and cube dimensions satisfy strict algebraic relations:

- Case C_1 (non-rotated cube): the pyramid base side length L equals its height H with one rational solution, implicitly consistent with golden-ratio proportions.
- Case C_2 (cube rotated 45° about the pyramid's axis): the ratio of pyramid to cube dimensions satisfies a quartic (or cubic) equation consistent with known "plastic constant".

Configuration 1

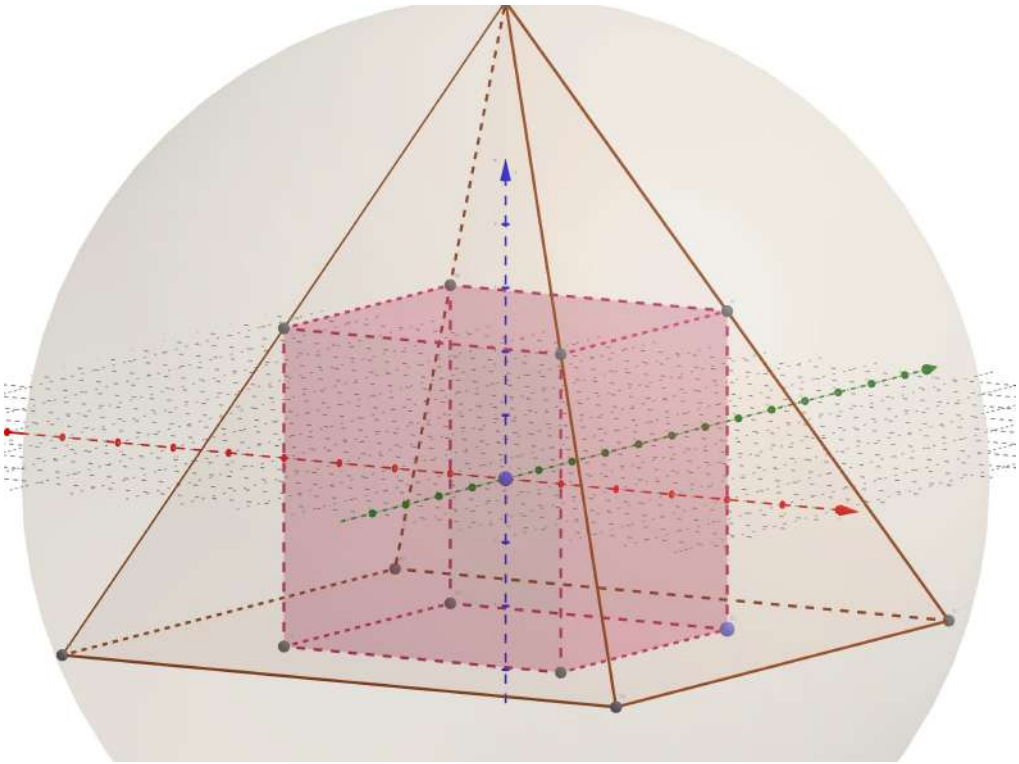

Figure 1. *Right square pyramid and central cube with cube base flush in the pyramid base as well as with centers coincidence and axis alignment*[1]

C_1. Algebraic Framework

To determine the relation between the cube of side k and the hosting pyramid with coincident centers of circumscribed spheres, we place the cube so that its base lies centrally on the pyramid's square base, and its top vertices touch the sloping faces of the pyramid (C_1).

The key is to express the pyramid's height H and base side L in terms of k, using the coincidence

1 Graph, Appendix A included, due to *GeoGebra*.

condition: the circumscribed sphere of the pyramid has the same center as that of the cube. This geometric constraint gives rise to a quartic equation for the pyramid's circumscribed radius r in terms of k (details in Appendix B: Algebraic Outlines). Symbolically, the relation reduces to the following quartic:

$$16r^4 - 16kr^3 - 8k^2r^2 - 4k^3r - 3k^4 = 0. \text{ (Q)}$$

Factorization over rationals (in symbols r and k) gives:

$$(2r - 3k)(k + 2r)(k^2 + 4r^2) = 0.$$

This equation admits rational solutions; only one of them is both positive and consistent with the geometry ($r > k/2$). Solving yields a proportional relation between r and k, and hence between the pyramid's dimensions and the cube

$$r = (3/2)k.$$

This solution shows that the pyramid dimensions are tied directly to the cube side length:

$$L = H = 2k.$$

For example, with $k = 10$ m, the acceptable root is $r = 15$ m, which gives $L = 20$ m and $H = 20$ m. This remarkably simple outcome shows that the pyramid is regular in proportions, with height equal to base side length.

Thus, the algebraic framework reveals a hidden simplicity beneath the apparently complex quartic structure (Q), reinforcing the elegance of the coincidence principle in 3D.

C_1. Symbolic Coincidence and Golden Ratio

An unexpected connection emerges when one considers the slant height of the pyramid. With $L = H$, for the slant height

$$h = H\sqrt{5}/2,$$

which is directly related to the golden ratio φ; explicitely

$$h = H(\phi - 1/2).$$

Thus the cube–pyramid coincidence not only satisfies rational proportions but also encodes one of the most remarkable constants of mathematics, long revered in art, architecture, and philosophy.

What we've landed on *behaves like an archetype*: a clean, almost "Platonic" configuration that sits at the intersection of symmetry, constraint, and coincidence [2]. In the plane, so called Coincidence Principle with triangles already captured this "sweet spot". In 3-D, the cube-in-pyramid coincidence seems to be the natural extension. A few observations that align with our thinking:

- **Craft intuition vs. formal math**:
  Historically, builders, masons, and carpenters have used "whole-number" ratios and geometric coincidences without ever codifying them as theorems. So a cube-in-pyramid with such neat ratios could easily appear as an "intuitive" design, but not as an explicit statement

in literature.

- **Hidden archetypes**:
  Like the golden ratio turning up in pentagonal symmetries, or Kepler's rhombic dodecahedron arising from cubes + pyramids [3, 4], these archetypes often get rediscovered when someone imposes a new constraint (here: "centers coincide").

C_1. Relation to CIS Triangle Configuration

This three-dimensional construct (C_1) mirrors the role of the CIS-triangle in two dimensions [5]. In this case, the governing law appears as quartic relation in the circumradius, with constraints collapsing to a simple proportionality. The cube–pyramid coincidence exemplifies how strict geometric constraints can give rise to elegant, unexpected proportionality. The rational solutions, coupled with hidden links to the golden ratio, suggests that such configuration may represent geometric archetypes. Beyond their intrinsic mathematical value, they may inspire new perspectives in design, architecture, and even symbolic thought.

Configuration 2

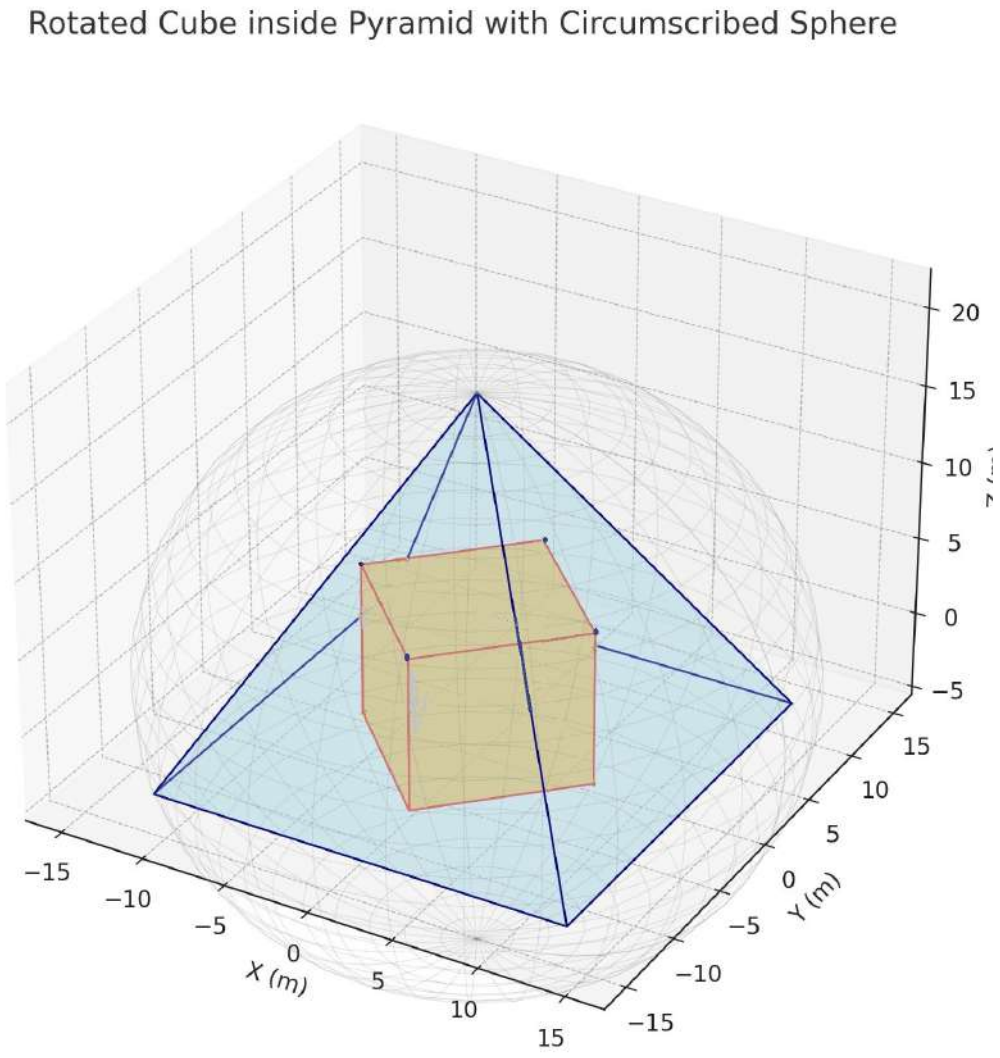

Figure 2. *The CIC pyramid with 45° rotated cube*[2]

2 Partly due to ChatGPT (by OpenAI, free).

C_2. Motivation and Analogy

Just as in the 2D framework where the CIS-triangle is complemented by the Λ-triangle [6], the 3D centrally inscribed cube (C_1) admits a rotated counterpart (although we have to be cautious - rotations, their orientations and meanings, are not quite the same for planar and spatial setups):

- Here the cube is rotated by 45° about the vertical axis of the pyramid.
- This ensures that four cube vertices now lie against the slant heights of the pyramid with the same base-to-base alignment.
- Centers of circumscribed spheres (cube and pyramid) remain coincident.

C_2. Algebraic Framework

In this part we outline the algebraic backbone of the axially rotated cube–pyramid coincidence (C_2) spatial configuration. Our goal is to express the relation between the cube edge k, the pyramid's base side L, its height H, and the radius r of the circumscribed sphere. The derivation is presented as a symbolic sketch, in line with the broader essay style, rather than a fully formal proof (otherwise, see Appendix B: Algebraic Outlines).

Starting from the two core relations for the rotated-cube (C_2) symmetric model: 1. Sphere (circumsphere) relation (distance from origin to base corner = r) and 2. Plane (tangency) relation (top rotated vertex lies on lateral face) we would finally reach

$$16x^4 - 16x^3 - 16x^2 - 12x - 5 = 0 \ (R)$$

with x = r/k. After polynomial factorization follows

$$(2x - 1)(8x^3 - 12x^2 - 2x - 5) = 0.$$

For a numerical model k = 10 m, solving the cubic, (R) gives the admissible root $x \approx 1.8247$, hence $r \approx 18.25$ m. From a general CIC-pyramid condition $H = r + k/2 \approx 23.25$ m and because (C_2) $L^2 = 2(r^2 + H^2)$, $L \approx 24.82$ m. A direct comparison with the C_1 setup reveals that same rotated cube here produces a larger pyramid in its basic dimensions.

C_2. Connection with the Plastic Constant

Another interesting connection surprisingly emerges[3]. Namely, if ρ is the (real) plastic constant [7, 8, 9] satisfying

$$\rho^3 - \rho - 1 = 0 \ (\rho \approx 1.324717957..)$$

then $x = \rho + 1/2$ is a root of (R). Immediately (expressed in closed form using ρ) follow:

$$r = k(\rho + 1/2),$$

$$H = k(\rho + 1),$$

and

$$L = k\sqrt{2}(\rho + 1)/\rho = k\sqrt{2}\rho^2$$

3 Inner symmetry firstly recognized by *Grok* system (xAI, free version), further emphasized by the author.

(details in Appendix B: Algebraic Outlines).

***

Dimensional Constants and Coincidence Geometry

The Coincidence Principle reveals a remarkable unity between geometry and algebra, expressed through proportional invariants that arise naturally from different dimensional alignments.
In the first configuration (C_1), the cube and pyramid share a central alignment without rotation. The resulting quartic r(k) relation simplifies to rational ratios linked to the golden ratio φ. This embodies the "planar" form of proportionality—characteristic of recursive geometries and two-dimensional harmony.

When the cube is rotated by 45°, as in the C_2 constellation, the algebra changes profoundly. The governing relations yield the plastic constant ρ, known from the cubic equation $\rho^3 - \rho - 1 = 0$. Here, the coincidence reflects a volumetric proportionality, extending the φ-type harmony into spatial (three-dimensional) expression. The appearance of ρ signifies a deeper algebraic symmetry rooted in cubic recursion.

The Λ-triangle, governed by $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ (also known as *tribonacci* constant [10]), stands as an oblique hybrid between these two. It exhibits proportional recursions that combine planar and spatial features, bridging the golden and plastic domains within a single geometric framework[4].

Taken together, φ, ρ, and λ represent a hierarchy of dimensional constants emerging from geometric coincidence: φ (planar), ρ (spatial), and λ (oblique hybrid). Each encapsulates a specific kind of symmetry, and their coexistence within the Coincidence Theorems points toward a unified algebraic structure underlying geometric harmony itself.

Uniqueness of the CIC: Archetype in 3D

The Centrally–Inscribed–Cube pyramid construction stands as a singular geometric archetype among all known polyhedral embeddings. It represents the one spatial realization where the Coincidence Principle achieves full closure—where the inscribed solids share not only the same center and circumscribed sphere, but also a precise and immutable network of contacts between their structural elements.

The CIC configuration restores uniqueness through constraint. The cube's vertices lie exactly on the pyramid's lateral edges, not merely within its volume. This single geometric requirement, when combined with the Coincidence Principle, overdetermines the system. The pyramid's proportions—its height, base, and circumscribed radius—resolve into one immutable ratio. No continuous family remains. The CIC is thus rigid: a single pyramid for each cube, and a single cube for that pyramid.

Compared to the free or rotated cube variants, whether by substitution of dual solids like octahedron or by alteration of base symmetry, cube-in-equilateral-triangle pyramid, and to the octahedral analogues, the distinction is clear. Those constellations admit a parametric continuum of realizations: shifting, sliding, or rotating within the same coincident sphere leaves their contact pattern unchanged. Only the CIC locks the solids into a fixed structural harmony, transforming coincidence from a geometric condition into a necessity. This archetypal uniqueness may thus be viewed as the three-dimensional consummation of the planar theorem—a point where coincidence ceases to be scaleless and becomes absolute.

4 Hence, the "λ connection" of the triangle with the Archimedes' snub cube [11] reveals a original spatial property.

Conclusion & Perspective

With all constructions and counter-attempts examined, the CIC pyramid resolves into a complete and closed structure. Only two genuinely distinct configurations—C_1 and C_2—satisfy the Coincidence Principle in a coherent three-dimensional geometry. Every other examined embedding fails to sustain coincidence without degeneracy. This final outcome grants the pyramid setup a privileged status: it embodies a point where geometrical necessity and spatial symmetry meet in a self-consistent equilibrium. Coincidence, in 3D, becomes an exception rather than a rule. In this sense the CIC pyramid does more than describe a construction; it discloses something deeper and more general about spatial order itself.

Having reached the closure of the three-dimensional Coincidence, the natural question now turns toward its possible transposition into higher dimensions. If the "Centrally Inscribed Cube" pyramid defines the last complete harmony within ordinary space, then awaits beyond it—where space itself unfolds into its own extension? The next step therefore naturally leads into *Hyperspace*, where coincidence and equivalence may acquire new meanings, unbound by the limits of our familiar geometry.

Acknowledgments

The author dedicates this work to the warm memory of Mr. Ranko Vurdelja, one of those quiets; splendid mind and noble soul. *To an exchange from messy crossroad of times and peoples.*

Notice on AI: Symbolic derivations were assisted by GPT-5 (OpenAI, free version), also checked with Grok (xAI, free version), used here partly as text formulation tool; main both algebraic and conceptual frameworks as well as interpretations are the author's own.

References

[1] Turanyanin, Dragan N. Theorem of Coincidence for Regular Polygons, *GS Journal*, (2025)

[2] Ghyka, Matila *Philosophie et Mystique du nombre* (1952); Serbian transl.: *Filozofija i mistika broja*, Novi Sad, (1988)

[3] Rhombic dodecahedron, *Wikipedia*

[4] Cardil, Roberto Kepler and the Rhombic Dodecahedron

[5] Turanyanin, Dragan N. The "Centrally Inscribed Square" Triangle, *GS Journal*, (2025)

[6] Turanyanin, Dragan N. Λ-Triangle, *GS Journal,* (2025)

[7] Plastic ratio, *Wikipedia*

[8] Piezas, Tito III; Ivan Lamoen, Floor; and Weisstein, Eric W. "Plastic Constant", From *MathWorld*--A Wolfram Resource.

[9] Spinadel, Vera W. de; Redondo Buitrago, Antonia, "Towards van der Laan's plastic number in the plane"(pdf), *Journal for Geometry and Graphics*, **13** (2), (2009)

[10] Weisstein, Eric W. "Tribonacci Constant." From *MathWorld*--A Wolfram Resource.

[11] Snub cube, *Wikipedia*

[12] Maier, Michael *Atalanta Fugiens*, Emblem XXI, Oppenheim (1618)

Appendix A: More graphs

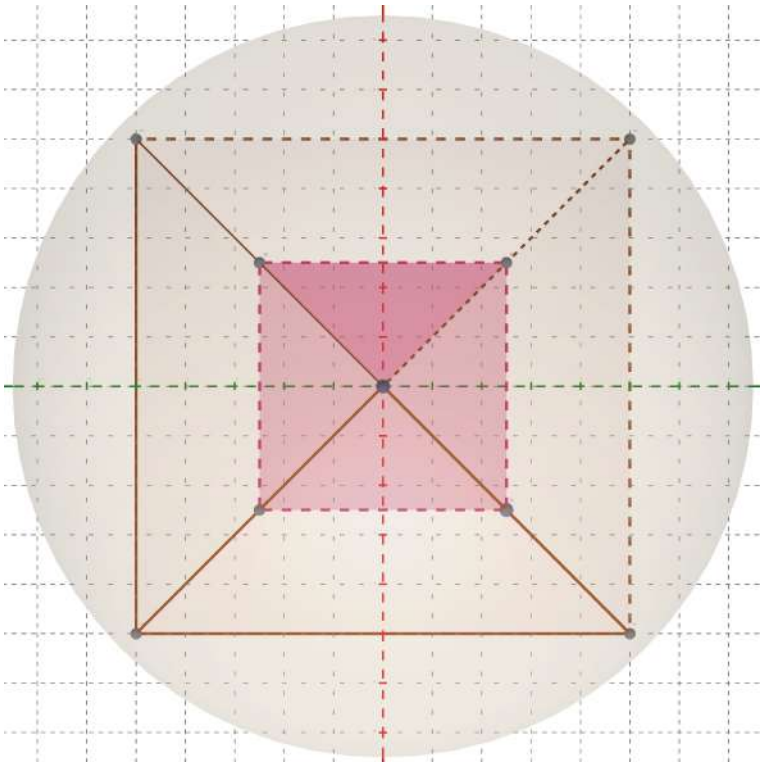

Figure 2. *CIC pyramid, C_1: Axial view*

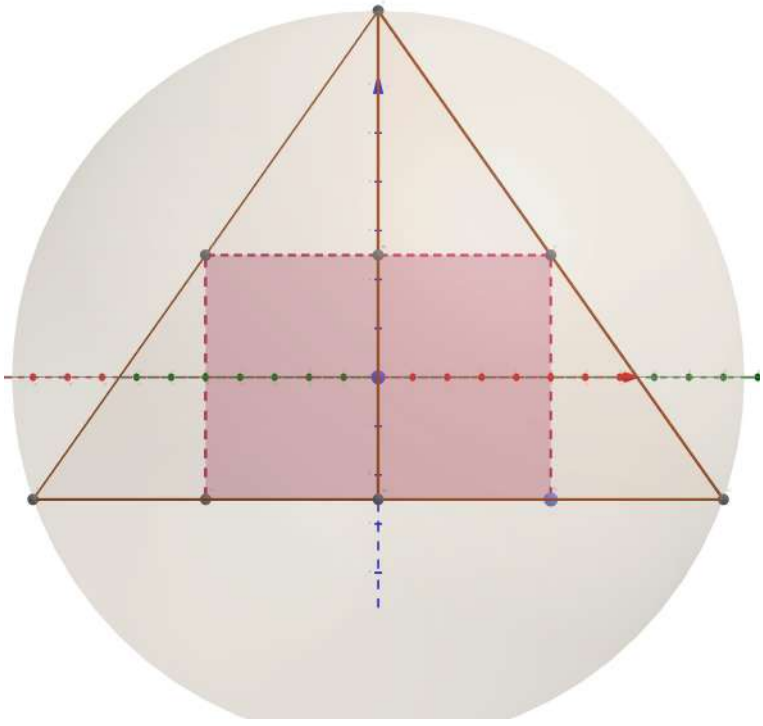

Figure 3. *C_1: Edge pyramid (cube diagonal) view*

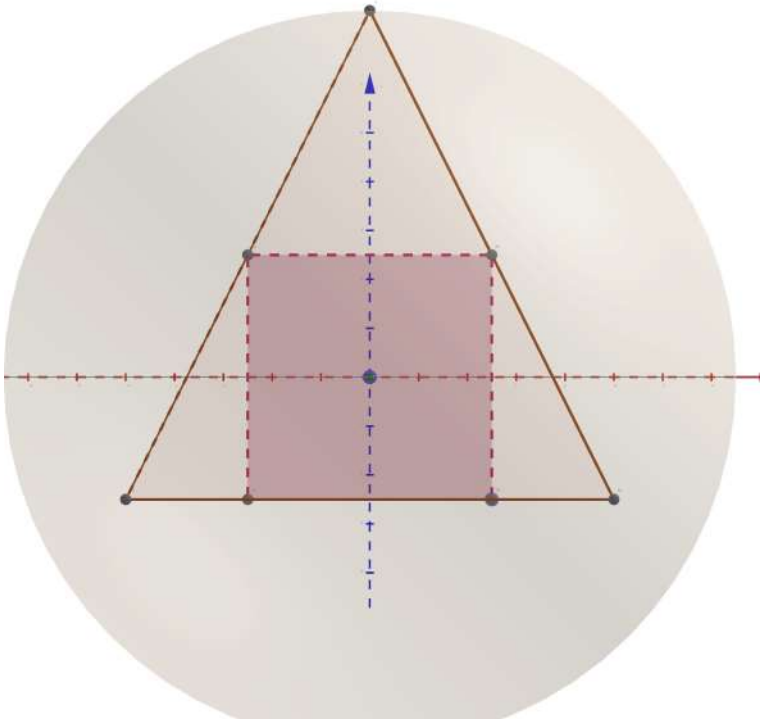

Figure 3. *C_1: Side view*[5]

Appendix B: Algebraic Outlines

Symmetric model and C_1 quartic

Setup (natural symmetric model):
• Let the cube have side k and be centered at the origin. Thus its bottom face lies in the plane z = -k/2 and the top face in z = +k/2.
• Consider a right square pyramid symmetric about the z-axis. The pyramid's square base lies in the same plane z = -k/2 so that the cube's bottom face is centrally placed on the pyramid base. The base has side L (half-side L/2).
• Let the pyramid apex be A = (0,0,r). Assume the common center of the cube and the pyramid circumsphere is the origin. Hence every base corner and the apex are at distance r from the origin (the circumsphere radius).

Geometric consequences (sphere/circumradius condition):
Distance from origin to a base corner (L/2, L/2, -k/2) equals r, so
$(L/2)^2 + (L/2)^2 + (-k/2)^2 = r^2$.

Solving for L^2 yields the basic relation
$L^2 = 2r^2 - k^2/2$. (1)

Plane/tangency condition (cube top vertex lies on lateral face):
Take the lateral face corresponding to the base edge with y = L/2. The face is the plane through the

5 The figure, at least symbolically, reminds of the Emblem XXI from M. Maier's *"Atalanta Fugiens"* [12]; also see Addendum: Alignment B, case C_1.

points
A = (-L/2, L/2, -k/2), B = (L/2, L/2, -k/2), and C = (0,0,r).

A convenient unnormalized normal vector is obtained from BA × CA; computing the dot product with (X - A) yields the plane equation.
Requiring that the (non-rotated) cube top vertex T = (k/2, k/2, k/2) lie on that plane gives a linear relation between L, r and k. Rearranging that relation produces the simple linear identity (equivalent forms):
$L(k/2 - r) = -k(r + k/2)$. (2)

This is the plane condition for the chosen lateral face and the non-rotated cube. Elimination (from linear relation to quartic):
To eliminate the base parameter L, square both sides of (2) to remove the sign ambiguity:
$L^2(k/2 - r)^2 = k^2(r + k/2)^2$. (3)

Now substitute L^2 from the sphere relation (1) into (3). This produces a single equation in r and k alone. Expanding and simplifying that equation yields the quartic polynomial in r (parameter k):

Quartic (Q) in r (as given in the manuscript):
$Q(r,k) = r^4 - (3/2)\cdot k\cdot r^3 + (5/4)\cdot k^2\cdot r^2 - (1/2)\cdot k^3\cdot r + (1/16)\cdot k^4 = 0$. (4)

Factorization and admissible root:
The quartic (4) factors over the rationals as a linear factor times a cubic:
$Q(r,k) = (2r - k)\cdot(8r^3 - 8kr^2 + 6k^2r - k)/16$.

Hence one rational root is r = k/2 (coming from the factor 2r - k). The remaining admissible root(s) are among the roots of the cubic factor; geometric admissibility (r > k/2, positivity, and L real and positive from (1)) selects the physically meaningful solution.

Remarks:
• The squaring step (2) → (3) can introduce extraneous algebraic roots; the factorization isolates those extraneous factors. In particular the linear factor (2r - k) corresponds to the rational root r = k/2 which must be checked against geometric inequalities (it may be degenerate or correspond to a limit case).
• The detailed intermediate algebra (expansion, clearing denominators, and factorization) can be carried out symbolically with a CAS; the final quartic and its factorization are displayed above.

Remarks on multiple rational roots:
• The algebraic elimination may produce several rational factors (e.g., 2r - k, 2r - 3k, etc.) depending on how denominators were cleared or which sign conventions were adopted during squaring.
• Each rational root must be checked against the geometric inequalities ensuring a positive height and real, positive L derived from (1).
• In this model, r = (3/2)·k is geometrically valid and consistent, while r = (1/2)·k corresponds to a degenerate configuration (the apex lying in the cube plane).
Thus the ratio r/k = 3/2 identifies the meaningful coincidence of the circumscribed sphere centers under the C_1 alignment.

L(k) C_1 derivation: cubic simplicity

The centers of the circumscribed spheres for the cube and pyramid coincide at (0, 0, k/2), assuming

coordinates where the pyramid base is centered at the origin on the z = 0 plane and the apex is at (0, 0, H).

The circumradius r of the pyramid satisfies two equations derived from distances to its vertices:
r = H - k/2,
r = √[L²/2 - (k/2)2].

The embedding condition requires the top edges of the cube to lie on the pyramid's lateral faces, leading to:
k = L(1 - k/H).

Solving this for H:
H = kL/(L - k).

Substitute into the expression for r:
r = kL/(L - k) - k/2.

Set equal to the other expression for r and let m = L/k (with m > 1):
(m + 1)/2(m - 1) = √(m2/2 +1/4).

Squaring both sides after simplification yields the cubic equation:
m3 - 2m2 + m - 2m = 0.

The real positive root is m = 2, so L = 2k and H = 2k. Thus,
L = H and r = 3k/2.

Cube–Pyramid C_2: axially rotated cube

This section presents the algebraic derivation for the rotated (C_2) configuration of a centrally inscribed cube within a square pyramid. This treatment yields a cubic in the normalized base length.

Symbolic Framework

Let k be the cube side. Define normalized variables x = L/k (pyramid base side over cube side), y = r/k (pyramid circumradius over cube side), and s = √2. The collinearity yields:
y = (1/2)(x + s)/(x - s). (1)

The sphere condition (pyramid base corner lies on circumscribed sphere) gives:
y² = x²/2 + 1/4. (2)

Eliminating y leads to the cubic:
x³ - 2√2·x² + 2x - 2√2 = 0. (P)

Alternatively, eliminating x yields the quartic in y:
16y⁴ - 16y³ - 16y² - 12y - 5 = 0. (Q)

Numerical Example (k = 10 m):
Solving the cubic (P) gives the admissible root x ≈ 2.482, so L ≈ 24.82 m. From (1), y ≈ 1.824, hence r ≈ 18.24 m. The pyramid height is H = r + k/2 ≈ 23.24 m.

Connection between C_2 - solution and the Plastic Constant

We work in the rotated-cube (C_2) setting using dimensionless ratio y = r/k (where r is the circumsphere radius/apex z-coordinate, and k is the cube side).

From the C_2 elimination we obtain the quartic equation in y:
$16y^4 - 16y^3 - 16y^2 - 12y - 5 = 0$. (Q)

Claim: If ρ is the (real) plastic constant satisfying $\rho^3 - \rho - 1 = 0$ ($\rho \approx 1.324717957..$), then
$y = \rho + 1/2$
is a root of (Q).

Proof (algebraic substitution): set $y = \rho + 1/2$ and expand (Q) in ρ. A short symbolic computation gives:
Symbolic reduction:
$16(\rho + 1)(\rho^3 - \rho - 1)$.

Thus (Q) becomes zero whenever $\rho^3 - \rho - 1 = 0$ (excluding the trivial $\rho = -1$ leading to $y = -1/2$ which is geometrically inadmissible).
Therefore the real plastic constant ρ yields the admissible C_2 root
$y = r/k = \rho + 1/2$.

Immediate corollaries (expressed in closed form using ρ):
• $r = k(\rho + 1/2)$,
• $H = r + k/2 = k(\rho + 1)$,
• L/k from sphere relation: $(L/k)^2 = 2(\rho + 1/2)^2 - 1/2$.

Simplifying using $\rho^3 = \rho + 1$ gives a compact form:
$L/k = \sqrt{(2\rho(\rho + 1))} = \sqrt{(2\rho*\rho^3)} = \sqrt{(2)}\rho^2$.

Hence the rotated-cube C_2 solution is expressible in terms of the plastic constant ρ as:
$r = k(\rho + 1/2)$, $H = k(\rho + 1)$, $L = k\sqrt{(2)}\rho^2$.

Numerical check ($\rho \approx 1.324717957..$):
$r/k \approx 1.0$

Conclusion: The plastic constant appears naturally in the rotated (C_2) coincidence configuration — the real root ρ of $\rho^3 - \rho - 1 = 0$ generates the exact closed-form relations between r, H, L and the cube side k shown above.

Addendum

Comparative Table: Six Cube–Pyramid Configurations

For clarity and reference, the following table summarizes six relevant geometric configurations of the cube–pyramid coincidence. Each configuration represents a distinct mode of central or boundary alignment between the cube and the pyramid—either non-rotated (C_1) or rotated by 45° (C_2)—and their corresponding positional relations of the circumscribed sphere's center.

Configurations considered: three circumsphere-center alignments (A: center of cube; B: center of top face; C: center of bottom face),

and two cube orientations for each alignment (C_1: non-rotated, C_2: rotated 45°).

Notation: k = cube side; L = pyramid base side; H = pyramid height (from base plane z=-k/2 to apex); R = circumsphere radius (distance from circumsphere center to apex).

Alignment A — circumsphere center at cube center (origin):
C_1 (non-rotated): r = 3/2 k, L = 2 k, H = 2 k, R = r = 3/2 k. (r is apex coordinate; H = r + k/2 = 2k)
C_2 (rotated 45°): r = k(ρ + 1/2), H = k(ρ + 1), L = k$\sqrt{2}$*ρ^2, R = r (ρ = plastic constant, $\rho^3 - \rho - 1 = 0$).

Alignment B — circumsphere center at cube top-face center (0,0,k/2):
C_1 (non-rotated): let t be root of $t^3 - 2\,t^2 + 3\,t - 4 = 0$, t ≈ 1.6506;
L = k t, H = k(1 + t)/(2 (t - 1)), R = k/(t - 1).
(Numeric example k=10: L≈ 16.506, H≈ 20.3697, R≈ 15.3697)
C_2 (rotated 45°): t solves $t^3 - 2\sqrt{2}\,t^2 + 4t - 4\sqrt{2} = 0$ with t ≈ 2.187;
L = k t ≈ 2.187 k, H ≈ 2.331 k, R ≈ 1.831 k.
(Numeric example k=10: L ≈ 21.87, H ≈ 23.31, R ≈ 18.31)

Alignment C — circumsphere center at cube bottom-face center (0,0,-k/2):
C_1 (non-rotated): L = k(1 + $\sqrt{2}$) ≈ 2.4142 k, H = k(1/2 + 1/$\sqrt{2}$) ≈ 1.2071 k, R = k(1 + 1/$\sqrt{2}$) ≈ 1.7071 k.
C_2 (rotated 45°): L = 2$\sqrt{2}$ k ≈ 2.8284 k, H = 3/2 k, R = 2 k.

November, 2025