New Perspectives on Conservation of Linear Momentum in Gravitational Fields

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Abstract

The absence of a conservation law for each component of linear momentum individually in a gravitational field does not mean that linear momentum is not conserved. The linear momentum in a gravitational field is conserved in the sum of its components. A body moving freely in a gravitational field does not exchange linear momentum with the gravitational field or with any other body; rather, the three components of its linear momentum exchange momentum among themselves in a manner that can be described by the law of conservation of angular momentum.

It is clear that we can define an energy for a body in a gravitational field—at least in the case of Schwarzschild spacetime—such that this energy remains constant during free motion. This is simply because spacetime in this case is symmetric with respect to time. Thus, the conserved form can be obtained either by using Noether's theorem and Killing vectors, or by a mathematical analysis based on General Relativity and general physical principles. In the same way, one can define an angular momentum calculated about the center of the gravitational source. Consequently, we obtain conservation laws for energy and momentum as follows:

$$\mathbf{E} = \mathbf{mc^2}\sqrt{\frac{1-\frac{2GM}{rc^2}}{1-\frac{v^2}{c^2}}}$$

$$\mathbf{L} = \mathbf{m}\frac{\omega r}{\sqrt{1-\frac{v^2}{c^2}}}$$

Now let us perform a formal change in the definition of energy in a way that does not affect its conservation nor the physical reality it describes, by squaring this quantity and dividing it by the mass. We shall later see the justification for using this alternative quantity. Thus, we now have the law of conservation of energy in another equivalent form:

$$\mathbf{E} = \mathbf{m}\frac{c^2-\frac{2GM}{r}}{1-\frac{v^2}{c^2}} = \text{Constant}$$

It is clear that it is not possible to find a definition of a conserved vector linear momentum for the body alone, due to simple experimental facts. For example, a body thrown upward returns to the same point with a speed equal in magnitude to that with which it was thrown, but opposite in direction. Likewise, the motion of planets around stars constantly changes its direction. These facts led most physicists to be satisfied with the laws of conservation of energy and angular momentum, without a conservation law for linear momentum, when dealing with motion in a gravitational field.

However, to anyone who examines this issue closely, this position involves a concession greater than the problem warrants. This problem can be solved at a lower cost by looking for a conservation law for the components of linear momentum when they are taken together, rather than separate conservation laws for each component individually, and this is something that can indeed be achieved. The definition of the conserved linear momentum is:

$$\mathbf{P} = \mathbf{m}\,\frac{v^2-\frac{2GM}{r}}{1-\frac{v^2}{c^2}} = \mathbf{m}\,\frac{(v_x^2+v_y^2+v_z^2)-\frac{2GM}{r}}{1-\frac{v^2}{c^2}} = \text{Constant}$$

We wrote the expressions in Cartesian coordinate system for simplicity and we will limit ourselves to it hereafter, although this analysis is valid for all types of coordinate systems.

The conservation of this quantity can be directly inferred from the law of conservation of energy. But this does not mean that it is a repetition or mere rephrasing of the conservation of energy. Rather, adding this definition and law leads to additional information for our understanding of the motion of bodies in a gravitational field, because this law is related to spatial coordinates collectively, whereas the law of conservation of energy is related to time.

This allows us to form an accurate picture of the momentum conservation law: in a gravitational field, the three spatial components of the energy–momentum vector are not conserved individually, but their total is conserved. *A body moving freely in a gravitational field neither loses nor gains momentum from outside (the gravitational field or other bodies); rather, the x,y,z spatial components of the momentum exchange momentum among themselves*, this becomes clear if we write each component of the linear momentum as follows:

$$\mathbf{P}_i = \mathbf{m}\,\frac{v_i^2-\frac{2GM\cos^2(r,x_i)}{r}}{1-\frac{v^2}{c^2}}$$

Where (r, x_i) is the angle between the radial distance and the coordinate x_i.

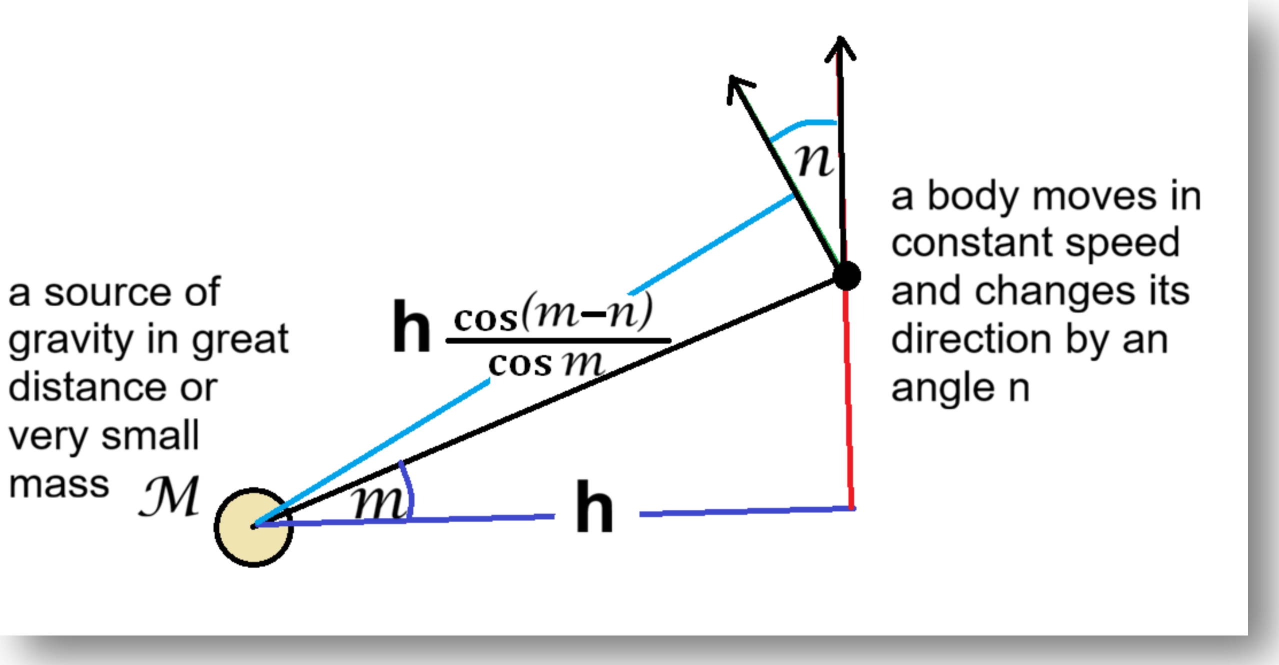
So that the conservation law of linear momentum takes the following form

$$\mathbf{m}\,\frac{v_x^2-\frac{2GM\cos^2(r,x)}{r}}{1-\frac{v^2}{c^2}} + \mathbf{m}\,\frac{v_y^2-\frac{2GM\cos^2(r,y)}{r}}{1-\frac{v^2}{c^2}} + \mathbf{m}\,\frac{v_z^2-\frac{2GM\cos^2(r,z)}{r}}{1-\frac{v^2}{c^2}} = \text{Constant}$$

What gives this fact great importance is that the details of the exchange of these momentum components among themselves are not chaotic or unknown, but controlled by the law of conservation of angular momentum. The conservation of angular momentum completely determines the direction of the velocity after the magnitude of the velocity is determined by the law of conservation of linear momentum, and thus the distribution of the conserved spatial momentum among the three spatial coordinates can be calculated.

We note that if we apply these two laws—namely, conservation of total linear momentum and conservation of angular momentum—in the case of the absence of a gravitational field, we obtain a conservation law for each of the three components, which is the well-known laws of conservation of momentum in special relativity, as follows:

The effect of the gravitational field depends directly on the value of the quantity$\frac{2GM}{r}$., If the gravitational source has a very small mass or is at a very large distance, this means that the magnitude of the body's velocity will not be affected when moving from one point to a neighboring point, because the value of $\frac{2GM}{r}$ remains approximately zero everywhere around the moving body. However, we note in this case that any change in the direction of motion of the body leads to a large change in the body's angular momentum; no matter how weak the gravitational source is. because any deviation from the straight line along which the body moves will affect the perpendicular distance h to the center of the gravitational source.



It is clear from the illustrative figure that if the body changes its direction by an angle n while moving from one point to a neighboring point, its angular momentum changes by a noticeable amount, determined by the factor $\frac{\cos(m-n)}{\cos m}$, and thus violating the conservation of angular momentum. Therefore, a body in absence or very weak gravitational field cannot change its direction by virtue of the law of conservation of angular momentum.

The constancy of the direction of motion of the body results in the constancy of the three components of the spatial momentum. Thus, the conservation of linear momentum in magnitude and direction existing in the absence of a gravitational field is nothing but a consequence of the law of conservation of total (scalar) linear momentum and the law of conservation of angular momentum.

Hence, it appears that searching for a conservation law for momentum components individually in a gravitational field, or believing that the absence of such a law implies a lack of information, is misguided. As we have seen, the law of conservation of linear total (scalar) momentum is the foundation, and when it is applied together with the law of conservation of angular momentum, it leads to a complete description of the directions of motion. The only thing is that this description, in the case of an extremely weak gravitational field, reduces to the conservation of momentum in each of the three spatial directions. Thus there is no fundamental difference regarding the law of conservation of linear momentum between the case of a gravitational field and the case of its absence.