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Mathematics that Every Physicist should Know: Scalar, Vector, and Tensor Fields in the Space of Real n-Dimensional Independent Variable with Metric

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Every physicist heard about vectors and tensors, but very few (if any) realize that they belong to the category of "numbers" -- carriers of a mathematical identity (like the real numbers do). It takes **n** or $\mathbf{n^2}$ real numbers to uniquely define a vector or tensor. These numbers (components) are loosing their number status because any single vector or tensor represent only one unique mathematical identity.

TENSOR ALGEBRA

Basic Concepts

Space of n-dimensional independent variable x^k with metric.

Space is a multitude of all points. Each **Point** P is defined in an Original Coordinate System K by the set of **n** ordered real numbers $\mathbf{x^1, x^2,... x^n}$, which are called "contravariant coordinates". Each coordinate takes values from minus to plus infinity. It is assumed that a "metric" (see below) is given in the original coordinate system K. According to the contemporary mathematical terminology it is "N-dimensional Geometry". We insist on the name "N-dimensional Theory of Field".

We are trying not to change the accepted meaning of mathematical dictionary terms and concepts except the concept of "Mathematical Object" which suppose to be a mathematical structure (e.g. a Group, Vector Space, or Differentiable Manifold) in a Category. Our definition goes as follows:

Mathematical Object is a carrier of **Mathematical Identity** (like single **Number**, single **Point**, single **Vector** (Vector with unique given components), or single **Tensor**). Note: coordinates of a Point, components of a Vector or Tensor are not math. objects; they are **Mathematical Accessory**. The transformation of coordinates (a particular law that allows us to manipulate coordinates and components of vectors and tensors), although can be very definite, is not a Math. Object in the Theory of Field because it manipulates Math. Accessory (in the Theory of Groups it will be different).

Any particular Math. Object can be given a **Name** of our choice.

Each Math. Object can be connected to a specific Point of Space. The Math. Objects that are not connected to any point of space are called "Global Objects" meaning that they belong to the whole coordinate system K (these objects can not be thought of as out of the coordinate system K completely).

Any multitude of Math. Objects is also a Math. Object. A Math. Object that includes Math. Objects that connected to the different points of space can be called **Extended Math. Object.**

Point: Point P can be thought of as an n-dimensional number which becomes definite when n ordered real numbers are given:

$$P: \quad x^1, x^2, ... \quad x^n \qquad (1)$$

The real numbers x^k, k=1,... n are called **coordinates** of the point P. This n-d number (that we call "point") carries unique identity as does any number in mathematics. The upper index k is put here deliberately in superscript, and can be expressed by any letter. Any letter index can take one of the values: 1, 2,... n.

Space: Space is a multitude of all possible points. Further, we consider the n-d number "point" only as an independent variable. In this case, the above mentioned space can be called **Space of a real n-d independent variable.**

Function in space (field): It is a dependent variable (for example expressed by a real number) that can be expressed as:

$$f(x^1, x^2, \ldots \quad x^n) \equiv f(x) \quad (2)$$

It depends on all the coordinates (in other words on the point in space).

Transformation of coordinates (arbitrary): Let us consider a transformation of the coordinates of the point P:

$$x^{k'} = x^{k'}(x) \quad (3)$$

where $x^{k'}$ on the left (k=1,... n) are the new coordinates of the very same point P, and $x^{k'}(x)$ on the right are not coordinates, but are given arbitrary functions of the old coordinates x^k of the point P.

It is very important to note that we consider that the new coordinates $x^{k'}$ are the new coordinates of the very same point P. **In other words: we presume that the identity of the point P survives the transformation.**

Since the coordinates of the point P are arbitrary, we consider that every point in the space acquires a new coordinates $x^{k'}$. We actually changing the whole coordinate system K (where the coordinates of the point were x^k) to a new coordinate system K' (where the coordinates of the same point are $x^{k'}$).

The transformation (3) is arbitrary, but not completely arbitrary. Clearly, it has to have an inverse transformation (because we should be able to track the identity both ways):

$$x^k = x^k(x') \quad (4)$$

Important note: To complete the transformation we have to express the old coordinates of the point P which are the arguments of every field function (like (2), but this statement also applies to every vector or tensor field) through the new coordinates using the formula (4).

Taking partial derivatives from (3) we will get a matrix of n^2 derivatives $\frac{\partial x^{k'}}{\partial x^i}$ and taking partial derivatives from (4) we will get another matrix of n^2 derivatives $\frac{\partial x^k}{\partial x^{i'}}$. These can be different for different points in space if the transformation is not linear. The determinant of the first matrix we denote as d (called the Jacobian of direct transformation) is inverse to the determinant of the second matrix we denote D=1/d (called the Jacobian of inverse transformation). These Jacobians, being functions of the point in space, should not turn to zero or infinity anywhere in space. This is the condition on the functions (3).

Let us take the partial derivative from coordinate $x^{k'}$ against $x^{i'}$, we have:

$$\frac{\partial x^{k'}}{\partial x^{i'}} = \delta_i^k = \frac{\partial x^{k'}}{\partial x^1}\frac{\partial x^1}{\partial x^{i'}} + \frac{\partial x^{k'}}{\partial x^2}\frac{\partial x^2}{\partial x^{i'}} + \ldots \quad + \frac{\partial x^{k'}}{\partial x^n}\frac{\partial x^n}{\partial x^{i'}} \equiv \frac{\partial x^{k'}}{\partial x^a}\frac{\partial x^a}{\partial x^{i'}} \quad (5)$$

where δ_i^k=1 if i=k, and zero otherwise. In (5) we have introduced a very convenient simplification: if some index (index a in our case) occurs 2 times in some expression, then that means the summation of similar expressions with this index going from 1 to n (as you can see in (5)). We will use this simplification widely in future. The indexes in some expressions that do not repeat themselves can take any single value from 1 to n and must be the same in expressions on the both sides of the equal sign, and in every term as well (as in (5)).

By the transformation (3) we can find the new coordinates for every point of space. So, now we have two systems of coordinates K and K' for the points of the very same space (remember: points preserve their identity).

Let us consider, along with the point P, another point P1, the coordinates of which are: x^k+dx^k. It will be "close" point. From (3) and (4) we have:

$$dx^{k'} = \frac{\partial x^{k'}}{\partial x^a} dx^a; \quad dx^k = \frac{\partial x^k}{\partial x^{a'}} dx^{a'} \qquad (6)$$

According to the first formula, given n infinitesimal numbers dx^k (and knowing the transformation functions (3)) we can find n another infinitesimal numbers $dx^{k'}$. If we put the last ones into the right part of the second formula (and knowing the inverse transformation functions (4)) we will return to the original infinitesimal numbers dx^k. We will check that (because it is a very important step):

$$\frac{\partial x^k}{\partial x^{a'}} dx^{a'} = \frac{\partial x^k}{\partial x^{a'}} \frac{\partial x^{a'}}{\partial x^b} dx^b = \delta_b^k dx^b = dx^k$$

There is no doubt that the collection of infinitesimal numbers dx^k is unique, and therefore caries its identity (it is infinitesimal n-d number). In other coordinates K' this identity is expressed by $dx^{k'}$. If we consider that dx^k are given, and we want to find $dx^{k'}$, then we have two ways to do so:

1. Find the coordinates of the points P and P1 in the coordinate system K', and then subtract the coordinates.
2. Use the first formula in (6).

Let us examine the second way. All of vector and tensor algebra is based on this clever trick: let us replace the infinitesimal numbers dx^k in (6) by the n finite real numbers A^k (this will form yet another kind of n-d number -- we call it a **vector**, and the numbers A^k we call the components of a vector). Using the first formula we will obtain $A^{k'}$ (the components of the same vector in coordinates K'). Then, using the second formula in (6), we can return to A^k. The conservation of the identity of the vector is insured by the same algebra that we checked above. The operation (6) and the introduced vector A^k belong to the point P in space (because the derivatives are taken in point P). The introduced vector can be different in different points of space forming a vector field in the space.

But this is not all. Let us consider the transformation:

$$B_k' = \frac{\partial x^a}{\partial x^{k'}} B_a; \quad B_k = \frac{\partial x^{a'}}{\partial x^k} B_a' \qquad (7)$$

This is another transformation law to compare to (6) but the conservation of the identity of the ***covariant*** **vector** B_k also takes place (and can be checked). Accordingly, A^k is called a ***contravariant*** **vector** (notice the ambiguity: by writing A^k sometimes we mean the identity that the vector carries, sometimes we mean the whole set of components of the vector, and sometimes we mean a separate component with the specific index k).

Notice: The n-d number "point" is not a vector because its transformation laws are (3) and (4). The n-d number "contravariant vector" transforms according to (6), and the n-d number "covariant vector" transforms according to (7).

Reminder: Since the vectors are the fields in space (their components depend on the coordinates of the point P), to complete the transformation to a new coordinate system K' we have to:

1. Obtain the new components expressing them through the old ones using the formulas (6) or (7).
2. Since the old components are the functions of old coordinates, we have to express these old coordinates of the point P through the new ones using the formula (4).

It makes a sense to call the transformation of coordinate systems by the name "**one point transformation**". Although the point is "every point of space", all the described above algebraic operations are performed at the very same point of space before going to another point of space, where the whole process repeats. The concept "**transformation of coordinates**" in general is a very important one in mathematics and theoretical physics. At first it seems, that it is the law that allows us to get n real numbers $x^{k'}$ if another n real numbers x^k are given. The theory of groups takes over and studies the different identities of these transformation laws. But this is not all. It is also important how we use the obtained transformed real numbers. If we are changing the coordinate system (as above), then they are the coordinates of the very same point in a new coordinate system. But we can consider another scenario: mapping. The coordinate system remains the same, the original point P with original coordinates remains the same, but the transformed real numbers are the coordinates of another point in the space P1. This notion is used in the symmetry investigations of the **extended mathematical objects** (see below). It makes a sense to call these transformations by the name "**two points transformations**".

Scalar: Let us consider a quantity:

$$A^a B_a \equiv \Phi; \quad \Phi' = A^{a'} B_a' = \frac{\partial x^{a'}}{\partial x^b} A^b \frac{\partial x^c}{\partial x^{a'}} B_c = \delta_b^c A^b B_c = A^b B_b = \Phi \qquad (8)$$

where we used (5). The scalar in n-d space is expressed by a single real number; it depends on a point in space (scalar field), and it does not change with a coordinate transformation (we can say that a scalar is invariant). The scalar doesn't have to be a scalar product of a contravariant vector on a covariant vector. It can be set up independently from vector or other fields.

Tensors: A vector can be called "a first rank tensor" because it has only one "free index" ("free index" means index that is not "contracted" (repeated) inside the expression). We can consider a "second rank tensor" A^{ik} with two "contravariant indexes", or second rank tensor B^i_k with the first index contravariant, and the second index covariant. The transformation laws will be correspondingly:

$$A^{ik}{}' = \frac{\partial x^{i'}}{\partial x^a} \frac{\partial x^{k'}}{\partial x^b} A^{ab}; \quad B^i{}_k{}' = \frac{\partial x^{i'}}{\partial x^a} \frac{\partial x^b}{\partial x^{k'}} B^a{}_b \qquad (9)$$

One can add any number of contra- or covariant indexes, and consider different rank tensors because the transformation laws can be written easily by analogy with (9). All of these tensors are n-d numbers (they carry unique identities) and are the fields in the space of n-d independent variable. The number of components of a second rank tensor is n^2, for a third rank tensor it is n^3 and so on.

The main property of vectors (tensors) is the fact that their identity survives the arbitrary transformation of coordinates while their components change. To underline this fact we can call scalars, vectors, and tensors by the name **mathematical objects.** To this category we can also add points, and any multitudes of points (like lines, triangles, and so forth. These we call **extended mathematical objects**). The coordinates of the points, the specifics of the definitions of multitudes of points, the components of vectors and tensors we can call **mathematical accessory.**

Another way to underline the fact that vectors (tensors) carry their identity is to widen the meaning of the term "invariant" by stating that **vectors (tensors) are the invariants (it does not matter that their components are changing) with respect to an arbitrary transformation of coordinates of a point in n-d space**. The previous notion was that the vectors (tensors) are "covariant" while the scalars are "invariant". There, the word "covariant", in this meaning, was not used to serve as a counterpart to "contravariant". Its second meaning was that vectors, tensors, and even scalars "transform according to their status". It was presumed that the objects we are talking about have their definite status in the first place. For example, for the scalars "covariant" meant that they are invariant. For vectors it meant that their components transform accordingly. So, the term "covariant", in this sense, was a general term applied to all mathematical objects.

As an example, let us bring to mind Einstein's famous "postulate of general covariance":

"We shall be true to the principle of relativity in its broadest sense if we give such a form to the laws (of physics**) that they are valid in every such 4-dimentional system of coordinates, that is, if the equations expressing the laws are co-variant with respect to arbitrary transformations."**

In the new terms, this postulate will sound like this: **Physical reality should be reflected in physical theory only by the mathematical objects (or invariants) of a 4-d space. The physical laws are the relations between these objects (invariants).**

In this way we are guaranteed that the physical laws and the physical values will express the same identity in any coordinate system.

Metric: Remember the two close points in the space P and P1. We saw in (6) that the difference of the coordinates of these points dx^k form a contravariant vector as long as this difference stays infinitesimal. We can think about the measure of the distance between these points as a one infinitesimal real number (scalar) given by:

$$ds^2 = g_{ab}dx^a dx^b; \quad g_{ik} = g_{ki} \qquad (10)$$

where g_{ik} is a covariant symmetric second rank tensor (called a **metric tensor**) that is given as an attribute of the space of a real n-d independent variable. This metric tensor is a field in space, and it is completely up to us to prescribe its particular value.

We have to consider the coordinate system where the metric is given (K in our case) as the unique original (primary) coordinate system, and all those (K' in our case) that obtained by arbitrary transformation (3), as the secondary coordinate systems.

Let us consider the algebraic equation for an unknown second rank contravariant symmetric tensor Y^{ik} :

$$g_{ia}Y^{ka} = \delta_i^k \qquad (11)$$

If the determinant of the metric tensor (we denote g) is different from zero or infinity (we consider this as the requirement on g_{ik}), then the solution of (11) Y^{ik} is unique and we denote it g^{ik}. We say that we have obtained the contravariant components of the metric tensor.

Now, for every contravariant vector we can find covariant components, and for every covariant vector we can find contravariant components by:

$$A_k = g_{ka}A^a; \quad B^k = g^{ka}B_a \qquad (12)$$

These components definitely describe the same identity as original components. This "lowering" and "raising" of the indexes can be applied to any rank tensor also. If we want to make a scalar product of two vectors, or the norm of a vector, we do not need one vector given in contravariant components and another in covariant components (as in (8)):

$$A^a B_a = g_{ab}A^a B^b = g^{cd}A_c B_d \qquad (13)$$

Examples: 1. Let us consider n=3 and the given metric tensor field is:

$$g_{11}=1,\ g_{22}=1,\ g_{33}=1,\ g_{ik}=0, i\neq k;\quad g^{11}=1,\ g^{22}=1,\ g^{33}=1,$$
$$g^{ik}=0, i\neq k;\quad ds=\sqrt{(dx^1)^2+(dx^2)^2+(dx^3)^2} \tag{14}$$

[This] is the simplest metric, called the Euclidean metric. It is used in physical geometry. With this metric, the contravariant and covariant components of vectors and tensors coincide. The above described algebra is greatly simplified, and usually only this simplified algebra is known to students. If we use an arbitrary transformation, then, in the secondary coordinates, this metric will get complicated, and the privilege of simple calculation will disappear. Therefore, it is meaningful to ask the question: is there some special coordinate transformations that won't change the components of the metric (14)? It appears that translations, rotations around any axis, and reflection of any axis satisfy this requirement.

2. Let us consider n=4 and the given metric tensor field is:

$$g_{11}=1,\ g_{22}=1,\ g_{33}=1,\ g_{44}=-1,\ g_{ik}=0, i\neq k;\quad g^{11}=1,\ g^{22}=1,\ g^{33}=1,$$
$$g^{44}=-1,\ g^{ik}=0, i\neq k;\quad ds=\sqrt{\varepsilon\left[(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2\right]} \tag{15}$$

where $\varepsilon=\pm 1$ ensuring that the radical is real. This is "pseudoeuclidean metric". This one is used in special relativity. It also simplifies the calculations, but you still have to keep track of co- and contravariant components because for x^4 they differ in sign. Are there any special coordinate transformations that won't change the components of the metric (15)? It appears that any translations, rotations involving axes 1-2, 1-3, 2-3, reflections of any axis, and Lorentz transformations involving axes 1-4, or 2-4, or 3-4 satisfy this requirement.

Amazing tensor field δ_i^k: Let us consider a field of a second rank tensor covariant with respect to its first index and contravariant with respect to its second index. Let us assign to every point in space the Cronecker's symbol δ_i^k as a numerical value to the components of this tensor field. In transformed coordinates K' we have:

$$\delta_i^{k\,\prime}=\frac{\partial x^a}{\partial x^{i\,\prime}}\frac{\partial x^{k\,\prime}}{\partial x^b}\delta_a^b=\delta_i^k \tag{16}$$

We see that in K', the components of this tensor field are the same as in K. This is an amazing tensor field that we can know up front for every coordinate system.

Fully antisymmetric tensor field: We are now looking for another "amazing" tensor field that can be known up front in any coordinate system.

The covariant components of the metric tensor form an n by n matrix the determinant of which g satisfies the algebraic relation that is true for any determinant:

$$g\varepsilon_{ik\dots l}=\varepsilon_{ab\dots c}g_{ia}g_{kb}\cdots g_{lc};\quad g=\frac{1}{n!}\varepsilon_{ik\dots l}\varepsilon_{ab\dots c}g_{ia}g_{kb}\cdots g_{lc} \tag{17}$$

where $\varepsilon_{ik\dots l}$ n-d fully antisymmetric **symbol** (not tensor). This symbol equals 1 if the transposition of ik... l is even, equals -1 if the transposition is odd, and equals zero if any pair of the indexes take the same value. The indexes of a symbol we always write as a subscript. The contraction of repeated indexes is still assumed. One can check these algebraic formulas in the case n=2, or n=3. We can find the determinant of a matrix of contravariant components of a metric tensor g^{ik} using (11) and the fact that determinant of a matrix δ_i^k equals 1. We have:

$$Det(g^{ik}) = \frac{1}{g} \equiv G \quad (18)$$

These determinants definitely form a field in space. Are they scalars? Expressing all the components of the metric tensor in (17) through the ones in transformed coordinates (using the law of transformation of the metric tensor $g_{ik} = \frac{\partial x^{m}{}'}{\partial x^{i}}\frac{\partial x^{n}{}'}{\partial x^{k}} g_{mn}{}'$) we can find that:

$$g = d^2 g'; \quad g' = D^2 g; \quad G' = d^2 G \quad (19)$$

This determinant constitutes a field in space. Is this field a scalar field? No, this is not a scalar field because the transformation law is different. This prompted mathematicians to introduce a special category for the field of determinant of the metric tensor -- **pseudoscalar**. The determinant of the metric tensor cannot turn to zero in any point of space. That means that the determinant stays positive or negative throughout the space. Let ε_g explicitly denotes the sign of g. The same can be said about the Jacobian of the coordinate transformation, and we denote the corresponding sign by ε_d . The transformations with ε_d=-1 are called improper transformations. From the equations (19) we can conclude that coordinate transformation can not change the sign of the metrics determinant. That means that ε_g is not a subject of transformations, and $\sqrt{\varepsilon_g g}$ always will be a real number.

Assumption I: Let us always choose plus (+) sign in front of the root $\sqrt{\varepsilon_g g}$ in the original coordinates K (positive chirality coordinates), and choose the following transformation law for this root:

$$\sqrt{\varepsilon_g g'} = D\sqrt{\varepsilon_g g}; \quad \sqrt{\varepsilon_g G'} = d\sqrt{\varepsilon_g G} \quad (20)$$

This root can take a negative value in some secondary negative chirality coordinates if the corresponding transformation is "improper" (ε_d=-1). Notice, that the assumption (I) necessitates the existence of the "original" coordinate system (in order for the root to have a unique sign).

Now we can introduce a **fully antisymmetric tensor field** (also called discriminant tensor, or tensor Levy-Chvitta):

$$\eta_{ik\ldots l} = \sqrt{\varepsilon_g g}\,\varepsilon_{ik\ldots l}; \quad \eta^{ik\ldots l} = \varepsilon_g\sqrt{\varepsilon_g G}\,\varepsilon_{ik\ldots l} \quad (21)$$

We can check to see that the usual procedure of raising indexes in $\eta_{ik\ldots l}$ from (21) will bring us to $\eta^{ik\ldots l}$ from (21), and vice versa.

It looks like this tensor field is known up front in any coordinate system (provided the metric is known). Let us calculate the components of this tensor in K' by applying usual tensor transformation law:

$$\eta^{ik\ldots l}{}' = \frac{\partial x^{i}{}'}{\partial x^{a}}\frac{\partial x^{k}{}'}{\partial x^{b}}\cdots\frac{\partial x^{l}{}'}{\partial x^{c}}\eta^{ab\ldots c} = \varepsilon_g\sqrt{\varepsilon_g G}\,\varepsilon_{ab\ldots c}\frac{\partial x^{i}{}'}{\partial x^{a}}\frac{\partial x^{k}{}'}{\partial x^{b}}\cdots\frac{\partial x^{l}{}'}{\partial x^{c}}$$
$$= \varepsilon_g\sqrt{\varepsilon_g G}\,\varepsilon_{ik\ldots l}\,d = \varepsilon_g D\sqrt{\varepsilon_g G'}\,\varepsilon_{ik\ldots l}\,d = \varepsilon_g\sqrt{\varepsilon_g G'}\,\varepsilon_{ik\ldots l} \quad (22)$$

During this derivation we used (17) (written for Jacobian) and (20). The last expression in (22) corresponds to what we would expect from (21) assuming that this tensor is known up front. This tensor is also an "amazing" tensor field, and it is not a pseudotensor field (as was assumed everywhere before. This change is due to the assumption (I)).

The equation (17) is written as non-covariant algebraic equation (because the left and the right parts are nether vectors nor scalars). After the introduction of the discriminant tensor (21) we can rewrite (17) in a covariant form. For this we have to multiply the first equation by $\varepsilon_g\sqrt{\varepsilon_g G}$ and the last equation by G :

$$\eta_{ik\dots l} = \eta^{ab\dots c} g_{ia} g_{kb} \dots g_{lc}; \quad 1 = \frac{1}{n!}\eta^{ik\dots l}\eta^{ab\dots c} g_{ia} g_{kb} \dots g_{lc} \qquad (17a)$$

Notice, that the pseudo-scalar field g has disappeared. This works only for the metric tensor. For other determinants we still have to use (17).

The volume element for integration over n-d volume[:] Let us consider n infinitesimal vectors in a point P:

$$dx^k_{(1)}[dx^1,0,\dots 0],\quad dx^k_{(2)}[0,dx^2,\dots 0],\dots \quad dx^k_{(n)}[0,0,\dots dx^n] \qquad (23)$$

where the lower index in brackets indicates the number of the vector. The components are given in square brackets. Notice, that each vector has only one component different from zero. Let us calculate the scalar:

$$\eta_{ab\dots c}dx^a_{(1)}dx^b_{(2)}\dots dx^c_{(n)} = \sqrt{\varepsilon_g g}\,\varepsilon_{ab\dots c}dx^a_{(1)}dx^b_{(2)}\dots dx^c_{(n)}$$
$$= \sqrt{\varepsilon_g g}\,dx^1 dx^2\dots dx^n \equiv \sqrt{\varepsilon_g g}\,d\Omega \qquad (24)$$

This is the volume element for integration. It is a true scalar (it won't change sign after improper transformation because the radical (see (20)) and the dΩ will change sign simultaneously).

For every tensor of a rank more than one it is important the order of the indexes. We can call them 1-st index, 2-nd index, 3-d index,... and so on. The importance of this order can be demonstrated on a discriminant tensor. Suppose n=3 and we have two vector fields A_i and B_i. Consider two contractions: A_i with the second and B_i with the third indexes, or A_i with the third and B_i with second indexes of the discriminant tensor. We have:
$\eta^{ikl}A_k B_l = -\eta^{ikl}A_l B_k$.

TENSOR CALCULUS

Our scalars, vectors, and tensors are fields, meaning that they depend on the point P which is an independent variable. Suppose we have a vector A_i (or a second rank tensor A_{ik}) given in its covariant components. That means that we have n (n^2) functions of coordinates. We can take a partial derivative from any of them against any of the coordinates. Let $A_{i,l}$ ($A_{ik,l}$) denoted the partial derivative from component A_i (A_{ik}) against coordinate x^l. In this way we get a total of n^2 (n^3) numbers -- exactly as many as the components of a second (third) rank tensor. Is this really a second (third) rank tensor? Checking the transformation law shows us that it is not a second (third) rank tensor (we can not get these derivatives in transformed coordinates by applying the transformation law of a second (third) rank tensor). The partial derivative of a scalar is an exception: it constitutes a covariant vector. It is a great inconvenience that we can not include our derivatives in formulas and still expect these formulas to have an objective meaning (the partial derivative of a vector or tensor is not a mathematical object -- it is an accessory). Though, it can be found that:

$$A_{i|l} = A_{i,l} - \Gamma^a_{il}A_a, \quad A_{ik|l} = A_{ik,l} - \Gamma^a_{il}A_{ak} - \Gamma^a_{kl}A_{ia} \qquad (25)$$

transform as a second (third) rank covariant tensor. We always denote a partial derivative by a coma and a covariant derivative by a vertical line to distinguish between them. The numbers $\Gamma^{a}{}_{il}$ are, the so called, Christoffel's symbols. They are defined by the formulas:

$$\Gamma^{a}_{ik} = g^{ab}[ik;b], \quad [ik;b] = \frac{1}{2}(g_{ib,k} + g_{kb,i} - g_{ik,b}) \quad (26)$$

These symbols are symmetric with respect ik indexes. They are not tensors and, therefore, can not be found in new coordinates by transformation from old coordinates. They have to be calculated in new coordinates from the metric tensor and this metric tensor in new coordinates can be obtained only by transformation from the old (original) coordinates where the metric tensor was given.

It is clear from (26) that in Euclidean space (14) as well as in pseudoeuclidean space (15) these symbols are zero, and a covariant derivative coincides with a partial derivative.

As another example: Let us take 3-d Euclidean space (14) as original coordinate system with a given metric. Let us consider a transformation:

$$x^1 = r\sin(\theta)\cos(\varphi), \quad x^2 = r\sin(\theta)\sin(\varphi), \quad x^3 = r\cos(\theta)$$
$$x^{1\prime} = r, \quad x^{2\prime} = \theta, \quad x^{3\prime} = \varphi \quad (27)$$

The matrix of derivatives from the old coordinates against the new coordinates will be:

$$\begin{pmatrix} \sin(\theta)\cos(\varphi), r\cos(\theta)\cos(\varphi), -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi), r\cos(\theta)\sin(\varphi), r\sin(\theta)\cos(\varphi) \\ \cos(\theta), \quad -r\sin(\theta), \quad 0 \end{pmatrix} \quad (28)$$

The determinant of this matrix is $D=r^2\sin(\theta)$. This determinant turns to zero in the origin of coordinates r=0 and on the line $\theta=0, \pi$. That means that the new coordinates are not legal. Still, we proceed, calling these spherical coordinates by the name **singular coordinates.** The components of the metric tensor in the new coordinates we can obtain by the transformation law of covariant components of the second rank tensor:

$$g_{ik}{}' = \frac{\partial x^a}{\partial x^{i\prime}}\frac{\partial x^b}{\partial x^{k\prime}} g_{ab}, \quad g_{11}{}' = (g^{11}{}')^{-1} = 1, g_{22}{}' = (g^{22}{}')^{-1} = r^2$$
$$g_{33}{}' = (g^{33}{}')^{-1} = r^2\sin^2(\theta) \quad (29)$$

The rest of the components are zero. The calculation of Christoffel's symbols using (26) gives:

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \Gamma^1_{22} = -r, \Gamma^1_{33} = -r\sin^2(\theta), \Gamma^2_{33} = -\sin(\theta)\cos(\theta), \Gamma^3_{23} = ctg(\theta) \quad (30)$$

The rest of them are zero. These mathematical techniques are equivalent to the usual known differential operations in spherical coordinates:

$$grad(vector): \Phi_{,k} = \Phi_{|k}; \quad div(scalar): A^{k}{}_{|k}; \quad curl(vector): \eta^{ikl}A_{k|l} \quad (31)$$

We learn from the above that a simple metric in original coordinates like Euclidean (14) or psevdoeuclidean (15) is a blessing. If we use an arbitrary transformation to the secondary coordinates, then almost every step will be accompanied by heavy algebra. Therefore, arbitrary coordinates have mostly "conceptual" meaning. The actual calculations are usually done in original coordinates, or special symmetry coordinates (like spherical). But without the "concept" of arbitrary coordinates, we can not understand what is meant by vector or tensor.

3. Noether's Theorem

One-Point and Two-Points Transformations of Coordinates

The concept "**transformation of coordinates**" in general is a very important one in mathematics and theoretical physics. At first it seems, that it is the law that allows us to get 4 real numbers x'^k if another 4 real numbers x^k are given. The theory of groups takes over and studies the different identities of these transformation laws. Nothing wrong with that. But this is not all to the concept "transformation of coordinates". The theory of groups studies a "transformation engine" only. It is also important how we use the obtained transformed real numbers x'^k. If we are **changing the coordinate system**, then x^k are the coordinates of the point P in original coordinate system K, while x'^k are the new coordinates of the **very same** point in a new coordinate system K'. This use of transformations we call:

One-point transformation of coordinates. We consider an arbitrary (Jacobian should be different from zero or infinity) transformation of coordinates. In each point P we consider scalars (given by one real number), vectors (given by 4 ordered real numbers), second rank tensors (given by 16 ordered real numbers), and so on. The transformation laws for

the covariant and contravariant components of a vector (and a tensor) are different:

$$x'^k = x'^k(x^i);\ x^k = x^k(x'^i);\ A'^k = \frac{\partial x'^k}{\partial x^i} A^i;\ A'_k = \frac{\partial x^i}{\partial x'^k} A_i;\ T'^{\ k}_{i} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x'^k}{\partial x^n} T^{\ n}_{m} \qquad (1)$$

The metric tensor g_{ik} in arbitrary coordinates does not have its simple Lorentz form, but it is considered to be given in original coordinate system K. The components of a metric tensor undergo transformation as a second rank tensor. The transformation changes the coordinate system, but does not change the point in space, nor it changes the scalars, vectors, and tensors that connected to this point of space. A scalar does not change its value that expressed by a real number. The components of vectors and tensors change, but they describe the same vectors and tensors only in a new coordinate system. In a new coordinate system everything acquires a new description: The mentioned point is described by a new coordinates, the vectors and tensors are described by the new components which are obtained from the old components in old coordinate system by a definite transformation laws (see (1)). Although the point P is "every point of space", all the described above algebraic operations are performed at the very same point of space before going to another point of space, where the whole process repeats.

All these components of vectors and tensors originally are functions of old coordinates. The arguments of all functions (which are old coordinates) should be **expressed** (not replaced) through the new coordinates. The whole operation physically means that we are not changing a physical object, which was described originally in coordinate system K, -- we are changing the coordinate system and giving a new description to the very same physical object. The main property of vectors (tensors) is the fact that their identity survives arbitrary transformation of coordinate system while their components change. To underline this fact we can call scalars, vectors, and tensors by the name **mathematical objects.** To this category we can also add points, and any multitudes of points (like lines, triangles, and so forth. These last we call **extended mathematical objects**). The coordinates of the points, the specifics of definitions of multitudes of points, the components of vectors and tensors we can call **mathematical accessory.** There is a good rule in theoretical physics: physical objects can be described only by mathematical objects; mathematical accessory has only indirect relations to the physical objects.

A one-point transformation of coordinate system is essentially an "arbitrary" transformation, because its main purpose is to introduce scalars, vectors, and tensors and separate mathematical objects from mathematical accessory.

Two-Point Transformations of Coordinates (symmetry transformation). We can consider another scenario of using the transformed coordinates: mapping. The coordinate system remains the same, the metric tensor remains the same, the original point P with original coordinates remains the same, but the transformed real numbers are the coordinates of another point P1 in the same space. This notion is used in the symmetry investigations of the **extended mathematical objects**. Actually, we are making a second copy of a physical object, which was originally described in the coordinate system. We take the physical properties (that are described by scalars, vectors and tensors) from the point P1 and transfer them to the point P. Along with that we keep the physical properties that were originally in the point P safe. Since the point P is "every" point of space, the point P1 also will go over the whole space. This way we will create a dabble set of the physical properties in every point of space making a second physical copy that we later compare with the original physical object. Many questions arise as how it should to be done. The scalars, probably, can be transferred without problems, but vectors and tensors definitely require a special treatment. To account for that we consider a "change of the form" of the vector and tensor components. Usually one prescribes the same transformations of the vector and tensor components as in one-point transformation (without expressing the old coordinates through the new ones). But this should be justified because the other ways are also possible. The transformation of coordinates itself also should be "physically" justified.

It should be clear that a two-point transformation is specific to a physical object (not arbitrary) and can be called "symmetry transformation".

Incorrect use of E. Noether's Theorem

The Noether's theorem is a statement regarding two-point transformations of coordinates only. It allows us to obtain a specific conserving quantity if the Lagrangian of the physical system invariant (remains the same for both physical copies in the every point of space) on the specific "symmetry" transformation (two-point transformation of coordinates with specific rules as how to transfer the physical properties from one point to another). Noether's theorem can not be applied to one-point transformation of coordinate system because the Lagrangian is a true scalar and it is invariant with respect to "arbitrary" one-point transformation.

The misuse of Noether's theorem is its application to one-point transformations. Theory of Groups studies the "transformation engine" only, pretending that it does not matter how this engine applied. The famous case is: if we take 4-translations as the "engine", then in two-point transformation context the majority of physical Lagrangians won't be invariant. But in one-point transformation context all Lagrangians will be invariant. From that comes the claim that the conservation of energy and momentum are the consequences of "symmetries" (here, allegedly, the homogeneity of 4-d space) that underlay "the dynamical laws of nature".

The specific dynamical laws of nature are the consequences of the requirement of minimum action with a specific Lagrangian. The conservation of energy and momentum is a consequence of the dynamical laws. There exists a unique procedure that allows us to obtain the energy-momentum tensor from the Lagrangian. In the light of this the previous relation can be reversed: the conservation of energy and momentum is a requirement (as the minimum of action) and the dynamical laws are the consequences.

Prof. D. Gross concludes regarding the negative attitude towards the theory of groups:

"That this attitude has changed so dramatically over the last 60 years, so that today principles of symmetry are regarded as the most fundamental part of our description of nature".

$$x'^k = x'^k(x^i);\; x^k = x^k(x'^i);\; A'^k = \frac{\partial x'^k}{\partial x^i}A^i;\; A'_k = \frac{\partial x^i}{\partial x'^k}A_i;\; T'_i{}^k = \frac{\partial x^m}{\partial x'^i}\frac{\partial x'^k}{\partial x^n}T_m{}^n \quad (31)$$

The variation principle is the very basis of any physical theory. It states that the variation of the action integral should be zero:

$$S = \int \Lambda \sqrt{-g}\, d\Omega, \quad d\Omega = dx^0 dx^1 dx^2 dx^3, \quad \delta S = 0 \qquad (32)$$

It is important to notice, that here the Lagrangian Λ is a true scalar. It is not a pseudoscalar because the expression $\sqrt{-g}\, d\Omega$ is a true scalar. Suppose we have a Lagrangian that depends on field functions u_k their covariant derivatives $u_{k|l}$ and coordinates x^k. The requirement of minimum action with respect of a variation of the field functions gives the Euler's equation which is regarded as a dynamical law of nature:

$$\left(\frac{\partial \Lambda}{\partial u_{i|k}}\right)_{|k} = \frac{\partial \Lambda}{\partial u_i}; \quad \Lambda(x^m, u_k, u_{k|l}) \qquad (33)$$

Suppose we have a linear infinitesimal 2 points transformation (31) and suppose that our field functions change according to:

$$x'^k = x^k + X^k_\alpha \omega^\alpha; \quad u'_i = u_i + \Psi_{i\alpha}\omega^\alpha; \quad u'_{i|k} = u_{i|k} + \Psi_{i\alpha|k}\omega^\alpha; \quad \Psi_{i\alpha} = u_{i|k} X^k_\alpha + S_{i\alpha} \qquad (34)$$

where $X^{k□}$ are constants, while □ $_{i□}$ are functions, and $S_{i□}$ describes the "change of the form" of the functions u_i (we parameterize the change of the form with the same parameter as the change of coordinates). Let us consider the quantities:

$$\Xi^k_\alpha \equiv \frac{\partial \Lambda}{\partial u_{i|k}}(X^l_\alpha u_{i|l} - \Psi_{i\alpha}) - \Lambda X^k_\alpha \qquad (35)$$

We can calculate the divergence of these quantities directly taking into account (33) and (34). We have:

$$\Xi^k_{\alpha|k} = \frac{\partial \Lambda}{\partial u_i}\left(X^l_\alpha u_{i|l} - \Psi_{i\alpha}\right) + \frac{\partial \Lambda}{\partial u_{i|k}}\left(X^l_\alpha u_{i|lk} - \Psi_{i\alpha|k}\right) - \Lambda_{|k} X^k_\alpha =$$
$$= -\frac{d\Lambda}{d\omega^\alpha}; \quad since: \quad \Lambda_{|k} = \frac{\partial \Lambda}{\partial u_i} u_{i|k} + \frac{\partial \Lambda}{\partial u_{i|l}} u_{i|lk} + \frac{\partial \Lambda}{\partial x^k} \qquad (36)$$

The Noether's theorem states that if □ does not depend on some parameter then we have the corresponding conservation quantity. The meaning of the theorem as follows: If we have described some extended mathematical object in a coordinate system K and if we manage to find such a two point transformation (31) and corresponding transformation of the scalar, vector, and tensor fields connected to this extended mathematical object (34), that after this transformation the Lagrangian does not change (remains the same function of coordinates).....

4-Translation

Let us consider a translation:

$$X^k_i = \delta^k_i; \quad x'^k = x^k + \delta x^k \qquad (37)$$

where the 4 infinitesimal parameters $\omega\ \alpha$ embodied by the 4 quantities δx^k. Now raises the question: Does Λ depend on a particular coordinate x^k or it does not? In (36) we have not a partial but full derivative with respect a parameter. Even if partial derivative is zero, the full derivative won’t be zero! The Lagrange density is invariant with respect 1 point translation, but here we have 2 points translation. We are not able to get the linear momentum conservation from Noether’s theorem.