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The Failure of Maxwell's Equations Inside a Capacitor

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The electromagnetic potential A^k of classical electrodynamics (CED) must be continuous across any boundary of disruption. This discovery was made by the present author(1) in 2003 (see also: www.classed.us/ConventCED/ConventCED.html). We believe that electromagnetic field and electromagnetic potential belong to the different physical realms. In this article we will try to treat the problem of plane capacitor in potential realm as if electromagnetic field does not exist at all.

The important change in our approach comes from the previous findings. We found that on any disruption surface:

a) The tangent components of the electromagnetic potential A^k must be continuous across the surface. The component of potential that is perpendicular to the surface might have a jump.

b) The scalar product of the energy-momentum tensor on the normal to the surface (it is a vector) must be continuous across the surface.

$$T^{ia}N_a \quad continuous \quad (1)$$

where N_k is a normal to the surface. This requirement goes along with the energy-momentum conservation:

$$T^{ik}{}_{|k} = 0 \quad (2)$$

We think that the requirement (1) has a preference over the Maxwell's equations.

The capacitance is usually defined in electrostatics. Will it change if we apply an alternating potential to a capacitor? Will the current change? Let us consider the case of plane capacitor in a circuit of alternating current. Suppose that a grounded source (with negligeable internal resistance) of alternating potential $\Phi(0,t)=F_1\exp(i\omega t)$ is connected in series with a left plate of a capacitor C which is located at x1=0, the right plate is located at x2=a, and a grounded resistor R. The area of the plates is S. The conventional solution of this problem is:

$$C=\frac{S}{4\pi a}; \quad I=\frac{\Phi(0,t)}{R}\frac{i\sigma_0}{1+i\sigma_0}; \quad \sigma_0=\omega RC \quad (3)$$

The two components of potential are essential for the plane capacitor: the time component Φ and the x-component A. They satisfy to the equations:

$$\Phi'' - \frac{1}{c^2}\ddot{\Phi} = -4\pi\rho; \quad A'' - \frac{1}{c^2}\ddot{A} = -\frac{4\pi}{c}j; \quad \frac{1}{c}\dot{\Phi} + A' = 0 \tag{4}$$

The last equation is the Lorentz gauge for potential (as seen from the realm of electromagnetic field) which we assume as a law in the realm of potential and call it "potential conservation".

Suppose that the potential at the second plate appeared to be $\Phi(a,t)=F_2\exp(i\omega t)$ (F_2 can be complex, while F_1 is considered real). The solutions of the equations (4) will consist of waves going in different directions of the x axis:

$$\Phi = \Phi^+ + \Phi^-; \quad \Phi^+ = F^+ e^{i(\omega t - kx)}; \quad \Phi^- = F^- e^{i(\omega t + kx)}$$
$$A = \Phi^+ - \Phi^-; \quad E = -\Phi' - \frac{1}{c}\dot{A} = 0 \tag{5}$$

because $\omega=ck$ in vacuum. We have come to the conclusion that the electric field between the capacitor's plates is zero as a direct result of the potential conservation requirement. This result is in agreement with the requirement (b). Let us take the condition (1) into consideration. In the case of plane capacitor N_k has only x component different from zero. The requirement (1) reduces to the continuity of T^{1k} or T^{11} since all other components equal zero in this symmetry. That brings us to the continuity of E. The conclusion is: if E=0 outside the capacitor's plate then it should be zero also inside.

Solving the boundary conditions for potential we find:

$$F_1^+ = i\frac{F_2 - F_1 e^{ika}}{2\sin(ka)}; \quad F_1^- = i\frac{-F_2 + F_1 e^{-ika}}{2\sin(ka)}$$
$$\Phi = \frac{e^{i\omega t}}{\sin(ka)}\left(F_2\sin(kx) - F_1\sin k(x-a)\right);$$
$$A = \frac{ie^{i\omega t}}{\sin(ka)}\left(F_2\cos(kx) - F_1\cos k(x-a)\right) \tag{6}$$

The current according to the usual definition of a capacitor is:

$$I_1 = I_2 = C\frac{d}{dt}\left(\Phi(0,t) - \Phi(a,t)\right) \quad \textit{static capacitor} \tag{7}$$

(I_1 being the current on the left side of the capacitor, I_2 – on the right side). This definition is not local because $\Phi(0,t)$ and $\Phi(a,t)$ are the potentials of the plates that are in different space locations. Instead, we use another (local) definition. Let us consider a boundary between the vacuum and the metal plate. We can write the wave equation for the both sides of the boundary (see (8)). Noticing that $\Phi_{vac}=\Phi_{in}$ we can subtract both equations to find the charge density. Using the conservation of charge we can find the longitudinal current density:

$$\Delta\Phi_{vac} + \frac{\omega^2}{c^2}\Phi_{vac} = 0; \quad \Delta\Phi_{in} + \frac{\omega^2}{c^2}\Phi_{in} = -4\pi\rho, \quad \sigma = \frac{\omega kRS}{4\pi} = \sigma_0 ka$$

$$-4\pi\dot{\rho} = 4\pi div\vec{j} = div\nabla(\dot{\Phi}_{in} - \dot{\Phi}_{vac}); \quad \vec{j}_{long} = \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla\Phi_{in} - \nabla\Phi_{vac})$$

$$I_1 = \frac{i\sigma}{R\sin(ka)}\left(F_1\cos(ka) - F_2\right)e^{i\omega t}; \quad I_2 = \frac{i\sigma}{R\sin(ka)}\left(F_1 - F_2\cos(ka)\right)e^{i\omega t} \qquad (8)$$

The capacitor's formula can be considered the same but the currents in the laft and in the right parts of the circuit are different. The current in resistor R is $I_2 = \Phi(a,t)/R$. We have:

$$I_2 = \frac{\Phi(a,t)}{R} = \frac{F_2}{R}e^{i\omega t}; \quad F_2 = \frac{F_1 i\sigma}{\sin(ka) + i\sigma\cos(ka)}$$

$$\Phi(x,t) = F_1\frac{\sin k(a-x) + i\sigma\cos k(a-x)}{\sin(ka) + i\sigma\cos(ka)}e^{i\omega t}; \quad I_1 = -\frac{\sigma}{R}A(0,t)$$

$$A(x,t) = -iF_1\frac{\cos k(a-x) - i\sigma\sin k(a-x)}{\sin(ka) + i\sigma\cos(ka)}e^{i\omega t}; \quad I_2 = -\frac{\sigma}{R}A(a,t) \qquad (9)$$

This defines the potential in vacuum and the currents if F_1 , a, and S are given.

To make the description complete we have to find the charge on the plates. For that purpose we can imagine the currents multiplied on a Heaviside's step function. Using the conservation of charge equation we found:

$$\rho_1 = -i\frac{1}{\omega S}I_1\delta(x); \quad \rho_2 = i\frac{1}{\omega S}I_2\delta(x-a) \qquad (10)$$

Now we can check the inhomogeneous wave equation for potential (4). Let us do it on the left plate because the right plate is similar. Going inside the plate we can represent the potential, its first derivative, and second derivative against x by:

$$\Phi(x,t) = \Phi(0,t)\theta(-x) + \Phi(x,t)\theta(x); \quad \Phi'(x,t) = \Phi'(x,t)\theta(x)$$

$$\Phi''(x,t) = \Phi''(x,t)\theta(x) + \Phi'(0,t)\delta(x); \quad \Phi'(0,t) = \frac{ke^{i\omega t}}{\sin(ka)}\left(F_2 - F_1\cos(ka)\right) \qquad (10a)$$

where θ(x) is Heaviside's step function. Using the inhomogeneous wave equation (4) for potential and expression for the current (8), we see that our formulas are in agreement.

Let us check the inhomogeneous wave equation (4) for the vector potential inside the conducting plates of the capacitor . We found the particular solution of inhomogeneous equation on the condition that the vector potential does not depend on x inside the metal plate:

$$A_{1in} = -\frac{4\pi c}{\omega^2}j_1; \quad A_{2in} = -\frac{4\pi c}{\omega^2}j_2 \qquad (11)$$

The vector potential is perpendicular to the plate of the capacitor. According to our statement (a) it does not have to be continuous across the boundary. Still, using (6), (8), (11), and assuming $j_k = I_k/S$, we can see that the boundary conditions with (11) are satisfied.
We can calculate the power that is generated in the left side circuit by multiplying the real part of I_1 on the real part of Φ(0,t) and taking time average. The same can be done for the right side. Remarkably, both powers appeared to be the same and equal:

$$W = \frac{F_1^2}{2R}\frac{(\sigma)^2}{(\sin(ka))^2 + (\sigma\cos(ka))^2}$$

Let us see if our results are physically meaningful.

1. In the case R=0 we have: $I_1 = iF_1\frac{\sigma}{R}\frac{\cos(ka)}{\sin(ka)}; \quad I_2 = iF_1\frac{\sigma}{R}\frac{1}{\sin(ka)}$ (here and further I omit the factor exp(iωt) in the current).
2. In the case of infinite R we have: $I_1 = -iF_1\frac{\omega kS}{4\pi}\frac{\sin(ka)}{\cos(ka)}; \quad I_2 = 0$
3. In the case if a distance between the plates is small compare to λ the approximation gives:

   $I_2 \approx \frac{F_1}{R}\frac{i\sigma_0}{1+i\sigma_0} \approx I_1; \quad ka = 2\pi\frac{a}{\lambda}$ This formula coincides with (3) and indicates that the difference between I_1 and I_2 is in the second order of a small value ka.
4. Everything works well unless we go to statics. If the frequency goes to zero then σ also goes to zero. Both currents go to zero (which is reasonable), but the amplitude of the vector potential becomes infinite (see (9)). This automatically excludes the static case from the present theory.

If the frequency is not small then ka becomes not small. The condenser becomes more like transmitter and receiver. But in any case we consider S big compare to a^2 so that we can neglect the effects on the edge of the capacitor. Also we assume that the wire connection of the capacitor provides the same phase on the every part of the condenser plate. We can ask the question: what kind of waves go from the left plate (transmitter) to the right plate (receiver)? It is the longitudinal waves of scalar and vector potential. Let us go into the realm of electromagnetic field. We have:

$$E = -\Phi' - \frac{1}{c}\dot{A} = 0; \quad H = 0 \qquad (12)$$

We have come to the conclusion that there is **no electromagnetic field** between the transmitter and receiver and, consequently, **no energy flow**. If it is so between the condenser plates then it will be so in general between transmitter and receiver – this is the idea of N. Tesla. But this idea was buried long time ago. What do we have against it? Well, Maxwell's equations on the boundary. Let us write the Maxwell's equations in vacuum:

$$div\vec{E} = \frac{4\pi}{c}j^0 \quad (13)$$

$$rot\vec{H} - \frac{1}{c}\dot{\vec{E}} = \frac{4\pi}{c}\vec{j} \quad (14)$$

The symmetry of a plane capacitor only allows E_x=E to be different from zero between the plates and E, j^0, and j^x to be different from zero inside the plates. Above we considered a source of alternating single frequency potential. The conclusion that can be made from (14) and the symmetry consideration is that the electric field between the plates can not depend on time. We know that there is an alternating surface charge density on the capacitor's plate. If we enclose the part of the plate by some closed surface that goes across the plate then (according to (13)) the charge inside should be proportional to the flux of electric field through the surface. We can satisfy this by assuming the presence of the electric field inside the capacitor and absence of it outside the capacitor. But we have a confusion: this electric field can not depend on time. It is clearly seen here that the Maxwell's equations fail on the capacitor's plate. Still, it may be possible in statics that the constant electric field is present inside the capacitor?

Static Capacitor.

It is interesting to apply the same principles to the static capacitor. Suppose Φ_1 is the static potential of the left plate. The solutions of the wave equation will be:

$$\Phi = \Phi_1\left(1 - \frac{x}{a}\right); \quad A = \Phi_1\frac{ct}{a} + C_1; \quad E = 0 \quad (15)$$

where C1 is an arbitrary constant. From (15) we can see that A jumps on the plates (which is permissible according to the requirement (a)) and growths infinite with time. Using the wave equation on the boundary (like in (10a)) we can find the charge:

$$Q = C\Phi_1; \quad C = \frac{S}{4\pi a}; \quad W = \frac{1}{2}Q\Phi_1 \quad (16)$$

Now raises the question: where this energy is located? Both plates carry equal and opposite charge. The right plate is at zero potential. That means that all energy is at the left plate. Suppose that at some moment we connect the left plate to the ground with a very small resistor R1. In the case if R1C > a/c we have a reason to expect that the both currents will be the same (see the new definition of the longitudinal current from (8) and take into account the symmetry of the picture). The potentials of the plates and the relished power will be proportional to the values of the resistors. On the other hand, if R1C < a/c then the potential of the left plate will come to zero before the information can reach the right plate and change its potential. There will be a short moment when both potentials are zero. Where is the energy at that moment?

The energy formula in (16) has the form of a product of potential on charge. It comes from the old presentation of Classical Electrodynamics (CED). According to

www.classed.us/ConventCED/ConventCED.html this term is not present in the energy-momentum tensor. The energy is connected only to the electromagnetic field and the massive currents (not electric currents, but usually the massive currents accompany the electric currents). For the case of capacitor we can think of the presence of the electric field alongside the currents inside the metal plates. Since we found that the electric field between the plates is zero we are coming to the violation of (b) – formula

(1). What does it mean if one of the requirements of energy conservation is violated? That means that the boundary itself can be the source (or drain) of the energy. In the case of capacitor only T^{11} jumps on the boundary. That means that the boundary can be the source of a linear momentum T^{01}. This is the way how energy “disappears” at the transmitter plate and “appears” at the receiver plate without flowing over “the air”.

References:

1. Y. N. Keilman, Phys. Essays, **16**, #3, (2003).