CHEBYSHEV STATISTICAL PARTITION FUNCTION: A CONNECTION BETWEEN STATISTICAL PHYSICS AND RIEMANN HYPOTHESIS

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ABSTRACT: In this paper we present a method to obtain a possible self-adjoint Hamiltonian operator so its energies satisfy $\zeta(1/2+iE_n)=0$, which is an statement equivalent to Riemann Hypothesis, first we use the explicit formula for the Chebyshev function $\Psi(x)$ and apply the change x=exp(u), after that we consider an Statistical partition function involving the Chebyshev function and its derivative so $Z=Tr\{e^{-\beta H}\}$, from the integral definition of the partition function Z we try to obtain the Hamiltonian operator assuming that $H=p^2+V(x)$ by proposing a Non-linear integral equation involving $Z(\beta)$ and V(x), We also study the case of a Berry Potential with interactions up to first order $H=xp+\alpha W(x)$, in this case the integral can be linearized to get a Numerical evaluation of W(x).

For the case of the Partition function involving primes $\sum_n e^{-sp_n} = Z_1(s)$ we apply a similar method to find the associated Hamiltonian $Z_1=Tr\{e^{-sH_1}\}$ $H_1=p^2+Q(x)$, so "prime numbers" could be considered as the "Energy levels" of H_1
Through the paper we will use natural units, so the Planck,s constant and Boltzmann,s k will have the value $K_B=\hbar=1$

- *Keywords:* Chebyshev function, explicit formula, Statistical partition function Z, Hamiltonian operator, Prime Counting Function, Berry Potential.

1.Chebyshev function as a partition function of Statistical Mechanics:

The Chebyshev function has the definition:

$\Psi(x) = \sum_{n\le x} \Lambda(n)$ Where $\Lambda(n)$ is the Mangoldt function given by the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad \text{and} \quad , \Psi(n) - \Psi(n-1) = \Lambda(n) . \quad (1.1)$$

Then if we define the function:

$$\Psi_0(x) = \begin{cases} \Psi(x) - \frac{1}{2}\Lambda(x) & \text{when} \quad x=p^m \\ \Psi(x) & \text{otherwise} \end{cases} \quad \text{and} \quad \lim_{x\to\infty}\frac{\Psi_0(x)}{x} = 1 \quad \text{so} \quad (1.2)$$

$\Psi_0(x) \to x$ (asymptotic result)

Where p is a prime and m a positive integer this function has the next integral representation:

$$\Psi_0(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s} = x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}) \quad (1.3)$$

(Von Mangoldt formula for Chebyshev function) log here is the natural logarithm, and the sum $\sum_\rho \frac{x^\rho}{\rho}$ runs over all the Non-trivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ Re(s) >1 , a bound for the Chebyshev function using PNT is

$$|\Psi_0 - x| \le C\left(x\log(x)\exp\left(-\sqrt{\log x}\right)\right) \quad (1.4)$$

Now if we derivate respect to x at both sides and make the change of variable x=exp(u) we have:

$$\sum_\rho e^{\rho u} = e^u - \dot{\Psi}_0(e^u) - \frac{e^u}{e^{3u}-e^u} \qquad \dot{\Psi}_0(e^u) = \frac{d\Psi_0}{du}(e^u) \qquad \text{if RH is true then} \quad \rho_n = \frac{1}{2} + it_n$$

so the equality (1.3) becomes:

$$\sum_n e^{-\beta E_n} = Z(\beta = -iu) = e^{u/2} - e^{-u/2}\dot{\Psi}_0(e^u) - \frac{e^{u/2}}{e^{3u}-e^u} \qquad \beta = \frac{1}{T} \qquad (1.5)$$

we have called $t_n = E_n$ and $iu = -\beta \to u = i/T$ where "u" is a real parameter and T is the Temperature of a certain hypothetical system in Thermodynamic equilibrium, where we can consider the "imaginary" temperature T as a "Wick rotated" version (angle θ=π/2) so the partition function has a finite "sum", the partition function is related for example to the probability of "finding" a root with $|t_n|$=a, which belongs to the case of finding a "microstate" of the system with energy a or energy –a in the form:

$$P_{a,-a} = P_a + P_{-a} = \frac{2}{Z(u)}Cos(au) \quad \text{Which is a real number.} \quad u=u(T)$$

The main objective of our method is to get a Hamiltonian self-adjoint operator

$$\hat{H} = \hat{p}^2 + V(\hat{x}) \to \hat{H}\,|\Phi\rangle = E_n\,|\Phi\rangle \quad \text{Where} \quad \zeta(\frac{1}{2} + iE_n) = 0 \quad (1.6)$$

Note that the partition function is just the Trace of the exponential of certain Hamiltonian operator $Tr(e^{-\beta\hat{H}}) = Z(\beta) = Tr\{\hat{U}\}$ $\hat{U}\hat{U}^{+} = 1$ and that the "Energies" of this Hamiltonian will satisfy:

$\hat{H}\,|\Phi\rangle = E_n\,|\Phi\rangle$ and

$$\sum_n e^{-\beta E_n} = Z(\beta = -iu) = e^{u/2} - e^{-u/2}\dot{\Psi}_0(e^u) - \frac{e^{u/2}}{e^{3u} - e^u} = \sum_{n=-\infty}^{\infty} e^{-\beta E_n} \qquad (1.7)$$

Where $\Phi = \Phi_n(x)$ are the "Eigenfunctions" of the Hamiltonian operator H.

In classical and semi-classical Physics if the variables p and x can vary continuously the partition function is defined via the integral (considering a mono-atomic and one dimensional theoretical gas interacting under a potential V(x))

$$Z(\beta) \approx \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp.e^{-\beta\left(p^2 + V(x)\right)} \quad \text{(Partition function) (1.8)}$$

We could approximate the series defining Z(β) involving Chebyshev function by an improper integral, if we perform integration over the variable "p" we have the approximate Non-linear integral equation (Urysohn-Stieltjes Integral equation of first kind) for the potential V(x):

$$\left(e^{u/2} - e^{-u/2}\dot{\Psi}_0(e^u) - \frac{e^{u/2}}{e^{3u} - e^u}\right)\sqrt{\frac{u}{\pi}} \approx \int_{-\infty}^{\infty} dx \exp\left(iuV(x) + i\frac{\pi}{4}\right) \qquad (1.9)\ \ u>0$$

Where the first 2 terms of Z(u) : $e^u\left(1 - \frac{\dot{\Psi}_0(e^u)}{e^u}\right)$, making u=log(x) the term inside the parenthesis approaches 0 for big x since $\frac{d\Psi_0}{dx} \to 1$ (1.10)

If the potential V(x) is invertible so $(V \circ f)(x) = x$, f is differentiable and $V(\infty) \neq V(-\infty) = a$, V(-∞)=b , we could calculate the inverse of the potential V(x) in the form of a Fourier integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} du \frac{Z(u)}{\sqrt{u\pi}} e^{-iu\tau + i\pi/4} \approx W_c^b(\tau) f(\tau) \qquad \text{with} \qquad W_b^a(\tau) = \mathrm{H}(\tau - b) - \mathrm{H}(\tau - a) \qquad (1.11)$$

"Window Function" and $\mathrm{H}(x) = \lim_{\varepsilon \to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega \frac{e^{i\omega x}}{\omega + i\varepsilon}$ (1.12)

The integral involving Z(u) must be understood as:

$$\lim_{\varepsilon\to o^+}\frac{1}{2\pi}\int_{-\infty}^{\infty}du\frac{Z(u)}{\sqrt{u\pi+i\varepsilon}}e^{-iu\tau+i\pi/4} \qquad (1.13)$$

(For the real case we would use the Fourier Cosine transform with a Kernel $Cos\left(\frac{\pi}{4}\right)Cos(u\tau)$ on the interval (0,∞)).

So using the inverse function theorem $\frac{1}{f'(V(x))}=V'(x)$ (1.14)

Of course instead of summing over Energies giving the imaginary part of the roots of $\zeta(\rho)=0$ we could have made the sum over $-E_n$ so $-iu=-\beta\to u=-i/T$, getting the complex conjugate of (4) if combine both Integral equations we get:

$$Z(u)\sqrt{u}\frac{Cos\left(\frac{\pi}{4}\right)}{\sqrt{\pi}}\approx\int_0^{\infty}dxCos(uV(x)) \qquad (1.15)$$ Since $Tr\left\{e^{iu\hat{H}}\right\}=\sum_{m=-\infty}^{\infty}e^{iuE_n}\in R$

Or using the identity for Dirac delta $\pi\delta(x)x\approx Sin(Nx)$ $N\to\infty$ then for big u :

$$\int_{u_0}^{u}du'Z(u')\sqrt{u'}\frac{1}{\sqrt{2\pi}}\approx\pi\int_{-\infty}^{\infty}dx\delta(V(x)) \quad u\to\infty \qquad (1.16)$$

Then the operator that gives all the roots of $\zeta(1/2+iE_n)=0$ is given by the linear differential equation (Sturm-Liouville operator) :

$$-\frac{d^2\Phi}{dx^2}+f^{-1}(x)\Phi=E_n\Phi \qquad \zeta(1/2+i\hat{H})|\Phi\rangle=0 \qquad (1.17)$$

Where the Z , partition function involves the derivative of Chebyshev function, this function has the Mellin-type transform: $\mathrm{M}(-s)=-\frac{\zeta'(s)}{\zeta(s)}=s\int_0^{\infty}dx\Psi_0(x)x^{-s-1}$

Using the delta function $\frac{1}{\log(x)}\frac{d\Psi}{dx}=\sum_{p,p^\nu}\delta(x-p^\nu)$ $\nu>0$ (sum over primes and prime powers)

An asymptotic evaluation of the Partition function approximating the sum by an integral, for big u so we can apply “Stationary Phase principle” .

$$\sum_n e^{iuE_n}\to\left|\frac{2\pi^2}{V''(\sigma)u^2}\right|^{1/2}e^{i[V(\sigma)u+\pi/2]} \qquad V'(\sigma)=0 \quad ,|V''(\sigma)|\neq 0 \qquad (1.18)$$

As $u \to \infty$, depending on whether the potential V is Real or complex (so $\hat{H} \neq \hat{H}^{\dagger}$) , the sum will diverge or tend to 0

If we define the function $Z_u(\eta) = \int_{-\infty}^{\infty} dx e^{iuV(x)+iu\eta x}$,so F has first continous derivatives respect to the argument 'tau' and the position then we can find using integration by parts and differentiation respect to parameter 'tau' the differential expression :

$$Z_u(\eta) = \int_{-\infty}^{\infty} dx e^{iuV(x)+iu\eta x} = -iu\int_{-\infty}^{\infty} dx x e^{iuV(x)+iu\eta}\left(\frac{dV}{dx}+\eta x\right) \qquad (1.19)$$

$$\int_{-\infty}^{\infty} dx x^n e^{iuV(x)+iu\eta x} = \left(-\frac{i}{u}\frac{\partial}{\partial\eta}\right)^n Z_u(\eta) \qquad (1.20)$$

$$iu\hat{x}\frac{d\hat{V}}{dx}\left(-\frac{i}{u}\frac{\partial}{\partial\eta}\right)Z_u(\eta)+\frac{\partial}{\partial\eta}Z_u(\eta)+Z_u(\eta)=0 \qquad (1.21)$$

Where, we have made the replacement $x \to -\frac{i}{u}\frac{\partial}{\partial\eta}$ to form the differential equation, to solve it we should expand $x\partial_x V(x)$ into a truncated Taylor series .

If we impose the 'Trace condition' (1.7) to our differential equation (1.21) , expanding the derivative of potential (operator) :

$$x\frac{dV}{dx}\left(\frac{-i}{u}\frac{\partial}{\partial\eta}\right) \approx a_1\left(\frac{-i}{u}\frac{\partial}{\partial\eta}\right)+a_2\left(\frac{-i}{u}\frac{\partial}{\partial\eta}\right)^2+........ \qquad (1.22)$$

Thus , we can obtain the value of the $\{a_i\}$ solving a certain system of equations since if V(x) satisfies a certain integral equation , setting $\eta = \frac{\pi}{4u}$ $u \neq 0$ we have the initial value condition:

$$Z_u\left(\frac{\pi}{4u}\right) \approx Tr\left\{e^{iu\hat{H}}\right\} = \sum_{m=-\infty}^{\infty} e^{iuE_n} = e^{u/2} - e^{-u/2}\dot{\Psi}_0(e^u) - \frac{e^{u/2}}{e^{3u}-e^u} \qquad (1.23)$$

$u = u_j \neq \mathrm{klog(p)}$ j=1,2,3,............. , so solving (1.23) would allow us to compute a_n n=1,2,3,…… so integratin by parts we could get the solution (up to a certain constant):

$$V(x) \approx C + \sum_{i=1}^{N} a_n \frac{x^n}{n} \qquad (1.24)$$

2. The Berry Potential with interactions

Since the Von Mangoldt formula is correct, hence for every linear operator of Hilbert-Polya type we must find that the trace of the Unitary operator must be equal to the expression (1.7) , then for a Berry potential with interactions of the form:

$$\hat{H} = \hat{x}\hat{p} + \alpha W(\hat{x}) \qquad (2.1)$$

$$Tr\left\{e^{iu\hat{H}}\right\} \approx \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \, \mathrm{e}^{iuxp} \left(1 + iu\alpha W(x) + O(\alpha^2) + ...\right) \qquad (2.2)$$

Where we have expanded the complex exponential inside the Semiclassical expression for the Trace upto first order on the coupling parameter obtaining a linearized integral equation for the interacting term W(x) of the form:

$$Tr\left\{e^{iu\hat{H}}\right\} - \frac{2\pi}{|u|} = iu\alpha \int_{-\infty}^{\infty} dp W_F(up) \qquad W_F(up) = \int_{-\infty}^{\infty} dx W(x) e^{iuxp} \qquad (2.3)$$

So we could use Numerical Quadrature methods to obtain $\left\{W_F(p_i u_j)\right\}_{i,j=1,2,3,.....}$ from (2.3)

Where the Trace of the Unitary operator involving H is given before in (1.7) . Note that since (2.3) is a linear integral equation there is only one solution at first order in α for the interaction W(x)=W(-x) (its Fourier transform must be a pure imaginary number if we want (2.3) to be Real)

3.Possible application to prime numbers:

Now if we consider the next "Partition function" given by :

$$\sum_n e^{-sp_n} = s\int_{-\infty}^{\infty} dt \pi(t) e^{-st} = Z_1(s) \qquad \text{with} \quad \beta = s = \frac{1}{T} \qquad \pi(x) = \sum_p \theta(x-p) = \sum_{p\le x} 1$$

(3.1)

We can get the integral equality considering primes to be some "Energy" levels

$$s^{3/2}\int_{-\infty}^{\infty} dt \pi(t) e^{-st} = L\left\{D_t^{3/2}\pi(t)\right\} \approx \sqrt{\pi}\int_{-\infty}^{\infty} dx e^{-sQ(x)} \qquad H_1 = p^2 + Q(x) \qquad (3.2)$$

Solving to get the inverse of Potential Q(x) ($(Q \circ g)(x) = x$) we find, taking inverse Laplace (Bilateral) transforms to both sides:

$D_\tau^{1/2}\pi(\tau) \approx \sqrt{\pi} W_b^a(\tau) g(\tau)$ Where $D^{1/2}$ can be considered the half-derivative operator:

$$D_x^r f(x) = \frac{1}{\Gamma(1-\{r\})} \frac{d^{[r]+1}}{dx^{[r]+1}} \int_c^x dt f(t)(x-t)^{-\{r\}} \qquad (3.3)\quad x-[x]=\{x\}$$ setting r=1/2 , [x] is the "floor" function.

In that case the Hamiltonian H_1 is bounded below , so its expectation value will satisfy $\langle H_1 \rangle \geq 2$ (there is a grounds state that belongs to the smallest prime number p=2)

Using Abel sum formula, and the definition of the half-derivative (differintegral) operator applied to the Prime counting function can be related (but some constants) to the series over all the primes up to x in the form :

$$\frac{1}{1-\{1/2\}} \sum_{p \leq x} (x-p)^{1-\{1/2\}}$$ and $\pi(x)$, Using Abel sum formula. (3.4)

Unfortunately we can only obtain approximations for the potential in the form (Ramanujan) involving Möbius function:

$$\sqrt{\pi} W_b^a(\tau) g(\tau) \approx \frac{1}{D^{1/2}} \left(\frac{1}{\tau \log \tau} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \tau^{1/n} \right) \qquad (3.5)$$

For the set of 'Eigenfunctions' of H_1 and the prime counting function, we can write:

$$\hat{\pi}\left(\hat{p}^2 + \hat{Q}(x)\right) |\mathrm{P}_n\rangle = n \qquad (3.6)$$

4.Conclusions and final remarks:

A possible connection between RH and Operator theory proposed by Hilbert and Polya was to find a Self-adjoint or Hermitian operator so:

$$\frac{1}{2} + i\hat{T} \,|\, n\rangle = \frac{1}{2} + it_n \,|\, n\rangle \qquad t_n = \Im m[\rho_n] \qquad \text{or} \quad \zeta\left(\frac{1}{2} + i\hat{T}\right) |\Phi\rangle = 0 \qquad (3.1)$$

In case this $\hat{T}^\dagger = \hat{T}$ exists, then we would have proved RH, our purpose in this paper is to find this operator T, considering it's a Hamiltonian of a certain Quantum system, using the explicit formula for the Chebyshev function and differentiating respect to x:

$$1 - \frac{d\Psi_0}{dx} - \frac{1}{x^3 - x} = \sum_\rho x^{\rho-1}$$ setting $\rho = 1/2 + iE_n$ and x=exp(u) (3.2)

(defined for every positive x except the points $x = p^{\nu} \quad \nu \in Z^{+}$, p=prime), the values of u that satisfy $\lim_{T \to T_H^-} Z(u) = Tr\left\{e^{iu\hat{H}}\right\} = \infty$, with $|T_H| = 1/\upsilon\log(p)$ are related to the "Hagedorn Temperature" -(s) of the system, so the "Trace" Z(u) of the operator above diverges.

Then the explicit formula is just the partition function Z for a certain hypothetical system at an imaginary Temperature "T" in the Thermodynamic equilibrium, this Z is precisely the Trace of a certain operator involving the Hamiltonian $Z = Tr[e^{-\beta\hat{H}}]$, where the Eigenvalues of this Hamiltonian are the imaginary parts of the Non-trivial roots of Riemann Zeta function, after that using the integral representation for the Z we have found a Non-linear integral equations for the V(x) in the form:

$g(u) = \int_{-\infty}^{\infty} dx K(u, V(x))$ (3.5) K(xu) is a Cosine, Sine or a complex exponential

This integral equation is obtained considering that $H=p^2+V(x)$, a particle of mass ½ moving under the influence of a potential V, and integrating over variable p.

A commonly used Numerical method would be approximate the Integral equation by a set of equations:

$$g(u_j) \approx \sum_i C_i K(u_j, V(x_i)) w(x_i) \qquad \mathrm{H}_n(x_i) = \mathrm{H}_n(u_j) = 0 \qquad w(x) = e^{-x^2} \quad (3.6)$$

Where H_n are the Hermite Polynomials defined by $\mathrm{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

And for the delta function we use the approximation: $\delta(x) \approx \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$ (n>> 1)

From the Non-linear system of equations we get the values $V(x_i)$ to calculate the "shape" of the potential.

Or introducing a Dirac delta function , using the property $g(u) = \int_{-\infty}^{\infty} dx g(x)\delta(x-u)$, we can solve the Urysohn-Stieltjes Integral equation (5) by iterative methods:

$g(u) + \varphi_{k+1}(u) = \int_{-\infty}^{\infty} dx F(u, \varphi_k(x))$ (in general $g(x) = \varphi_0(x)$)

$$F(u, V(x)) = K(uV(x)) + \delta(x-u) \text{ so } \qquad \delta(x-u) \approx \frac{n}{\sqrt{\pi}} e^{-n^2(x-u)^2} \qquad (3.7)$$

To calculate the approximate integral equation we have used the "Semi-classical approximation" for the partition function Z (sum over energies)

$$Z(\beta)=\sum_n e^{-\beta E_n}\approx\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\exp(-\beta H(x,p))$$

With $Z(u)=e^{u/2}-e^{-u/2}\dot{\Psi}_0(e^u)-\dfrac{e^{u/2}}{e^{3u}-e^u}$ if u>0 , $u=i\beta$ (3.8)

So the "Eigenvalues" E_n of the Hamiltonian, for are precisely the roots $\rho_n=\frac{1}{2}+it_n$ using this "Semi-classical" WKB approach for n "big" and a potential V(x) then:

$$\int_\gamma dx\sqrt{E_n-V(x)}=(n+\frac{1}{2})\qquad \zeta(\frac{1}{2}+iE_n)=0 \text{ and } n\to\infty \quad (3.9)\quad (\gamma=\text{closed path})$$

Then the "energies" and imaginary parts of the Non-trivial zeros satisfy $\lim_{n\to\infty}\dfrac{t(n)}{E(n)}=1$

From the "physical" point of view for the partition function Z(u) ,describing "|u|" the inverse of the Temperature of the gas, the approximation of a sum over "Energies" by an integral considering that (x, p) vary smoothly, it gets better as we approach to $u\to 0$ (classical behavior), and for big "Energies".

Also studying the case of a Berry potential with interaction at first order W(x) we can deduce certain symmetry properties of the potential so $W(x)=W(-x)\in L^2(R)$ in the WKB approach for the Trace of the Hamiltonian operator.

If we wish to impose the condition that our Kernel is on a L^2 Space:

$$\int_0^\infty dx\int_0^\infty du\,|Cos(uV(x))|^2<\infty \text{ Or equivalently } \int_0^\infty du\,|Z(u)|^2\,u<\infty\ ,$$

$$\int_1^\infty dx\log(x)\left|\frac{d\Psi_0}{dx}-1\right|^2<\infty \qquad (3.10)$$

Then we could use the bound given in (1.4) for the expression x >1 $\left|d\Psi_0-1\right|$. Note that the condition above is not fulfilled if there are roots of the form $\rho_n=a+it_n$, a≠1/2 ,which make real exponentials of the form $\left\{e^{(1/2-a)u},e^{-(1/2-a)u}\right\}$ to appear in the formula for Z(u), so it diverges in the limit u→∞ (for u<0 the Z(u) is 0) .

If we take the inverse of the potential V(x) then the "complex version" of our integral equation (4) includes the term:

$$\int_a^b dx f'(x)e^{iux}\to 0 \qquad \text{as } u\to\infty \quad (\text{"Riemann-Lebesgue" lemma})$$

In case V(x) and f(x) are real, so we must have the limit, $\lim_{u\to\infty}Z(u)=0$

For the case of "prime numbers" the "Temperature" (s=1/T) is Real , and the associated Partition function is $\sum_n e^{-sp_n} = Z_1(s) = Tr\left\{e^{-s\hat{H}_1}\right\}$

In Both cases we have used as pointed before the approximation:

$$Z(\beta) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \exp(-\beta p^2 - \beta V(x)) \approx \sum_n e^{-\beta E(n)} \quad \beta^* = -\beta \quad \text{or} \quad \beta \in R^+ \qquad (3.11)$$

Although there are many examples of proporsed Hilbert-Polya operators involving Riemann Hypothesis, from the Von Mangoldt formula for Chebyshev function, we have deduced a condition that must be satisfied by every linear operator of Hilbert-Polya type , in the sense that the Trace of exp(iuH) must be equal to:

$$\sum_n e^{-\beta E_n} = Z(\beta = -iu) = e^{u/2} - e^{-u/2}\dot{\Psi}_0(e^u) - \frac{e^{u/2}}{e^{3u} - e^u} = \sum_{n=-\infty}^{\infty} e^{-\beta E_n} \quad (3.12)$$

with singular values at the log of primes and prime powers, otherwise we could find the next contradiction:

$$\zeta(a + i\hat{H}_a)\left|\Phi_n^a\right\rangle = 0 \qquad \zeta(1/2 + i\hat{H})\left|\Phi_n\right\rangle = 0 \qquad (3.13)$$

So, we have found 2 Hamiltonians mapping all the zeros with real part 'a' and real part ½ (Riemann Hypothesis) acting over a different space of functions with no common Eigenvalues, However (4.11) does clearly not imply RH but the fact that Hilbert-Polya Hypothesis can be true but that could be useless to prove RH unless you restrict your possible operators to these with a Trace given at (3.12), this 'constraint' on the Trace is of vital importance and should always be taken into account when proposing any operator solution to "Riemann Hypothesis".

Also the solution to the integral equation ,determined by function $Z_u(\eta)$, so for the value $\eta = \tan^{-1}(1)$ we recover again the Trace of e^{iuH} . Hence we can reformulate our problem of finding V(x) inside the Hamiltonian , either solving the integral equation (1.9) or a differential equation given at (1.21-23).

Considering "Stirling Approach" for the factorial, and supposing that Chebyshev gas is Mono-atomic, and Non-interacting so $V_{TOTAL}=V_1+V_2+ \ldots\ldots\ldots\ldots +V_N$ then the Partition function of the Whole system at T=T(u) is:

$Log Z_{TOTAL} = N(\log Z(u) - \log N + 1)$ From this we can define every Thermodinamical

quantity , related to the system for example: (in absolute value)

$|F| = \frac{\log Z_{TOTAL}}{u}$ and $|U| = \frac{\partial}{\partial u}\log Z_{TOTAL}$ "Energy" and "Helmholtz function"

The Semi-classical approach (in natural units and with a Normalization factor) reads:

$$Z_{TOTAL} = \frac{1}{N!}\int_V dPdXe^{-\beta H(x_1,x_2,......,x_N,p_1,p_2,.........,p_N)}$$ V=volume of phase space.

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