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ZETA REGULARIZATION APPLIED TO THE PROBLEM OF RIEMANN HYPOTHESIS AND THE CALCULATION OF DIVERGENT INTEGRALS

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ABSTRACT: In this paper we review some results of our previous papers involving Riemann Hypothesis in the sense of Operator theory (Hilbert-Polya approach) and the application of the negative values of the Zeta function $\zeta(1-s)$ to the divergent integrals $\int_0^\infty x^{s-1}dx$ and to the problem of defining a consistent product of distributions of the form $D^m\delta(x)D^n\delta(x)$, in this paper we present new results of how the sums over the non-trivial zeros of the zeta function $\sum_\rho h(\rho)$ can be related to the Mangoldt function $\Psi_0(x)$ assuming Riemann Hypothesis.Throughout the paper we will use the notation $\zeta_R(s)=\zeta(s)$ meaning that we use the zeta regularization for the divergent series $\sum_{n=0}^\infty n^s$ s>0 or s=0

- *Keywords:*Zeta regularization, Urysohn equation , exponential nonlinearity , Riemann Hypothesis Hilbert-Polya operator, divergent integral

1.Spectral Zeta function $\zeta_H(s)$ and Riemann Hypothesis :

In case Riemann Hypothesis (RH) is true, in a previous paper [6] we give the physical equivalence between the explicit formula for the Chebyshev function $\Psi_0(x)$ and the formula for the trace of the Unitary operator $\hat{U}=e^{iu\hat{H}}$, where H is the Hamiltonian operator $\zeta\left(\frac{1}{2}+i\hat{H}\right)|\phi_n\rangle=0$, that is H is precisely the Hilbert-Polya operator solution to Riemann Hypothesis , let be the integral representation

$$\Psi_0(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s} = \begin{cases} x - \sum_\rho \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1-x^{-2}) & \mathrm{x} > 1 \\ 0 & \mathrm{x} < 1 \end{cases} \qquad (1.1)$$

Letting $x = e^u$, and differentitating with respect to 'u' we find the (trace) identity

$$e^{u/2} - e^{-u/2}\frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u}-e^u} = \sum_{n=-\infty}^{\infty} e^{iuE_n} = Tr\left\{\hat{U} = e^{iu\hat{H}}\right\} \qquad \mathrm{u} > 0 \qquad (1.2)$$

Using the semiclassical representation for the trace $\sum_{n=-\infty}^{\infty} e^{iuE_n}$ in terms of an integral over Phase Space , we have that the potential V(x) inside Hamiltonian H can not be arbitrary but must satisfy a kind of nonlinear Urysohn integral equation (r > 1)

$$\int_{-\infty}^{\infty} r^{iV(x)}dx = \sqrt{\frac{\log(r)}{\pi r^{-1}}}\left(1 - \frac{d\Psi_0(r)}{dr} - \frac{1}{r^3 - r}\right)e^{-i\pi/4} \qquad r = e^u \qquad (1.3)$$

The derivative of the Chebyshev function is defined as $\frac{d\Psi_0(x)}{dx}\frac{1}{\log(x)} = \sum_{p,p^\nu}\delta(x - p^\nu)$ (sum taken over prime and prime powers) .However (1.3) is too complex to have a known analytic solution, a good method to solve would be to suppose that the Operator proposed by Berry and Keating [2] plus an interaction is the correct Hilbert-Polya operator, in that case $H_b = xp + \alpha W(x)$ and we can linearize (1.3) at first order in the coupling constant 'α' as

$$Tr\left\{e^{iu\hat{H}}\right\} - \frac{2\pi}{|u|} \approx iu\alpha\int_{-\infty}^{\infty} dp\hat{F}\left\{W(x),u\right\} \qquad \hat{F}\left\{W(x),u\right\} = \int_{-\infty}^{\infty} dx e^{iuxp}W(x) \qquad (1.4)$$

Also, if we introduce the function $Z_u(\eta) = \int_{-\infty}^{\infty} dx e^{iu(V(x)+x\eta)}$, with continuos partial derivatives $\partial^k{}_\eta Z_u(\eta)$, then solving (1.3) is equivalent to finding a solution to the initial-value problem

$$Z_u(\eta) + \eta\frac{\partial Z_u(\eta)}{\partial\eta} + iu\left(\sum_{k=0}^{\infty} d_k \frac{k}{(iu)^k}\frac{\partial^k}{\partial\eta^k}\right)Z_u(\eta) = 0$$

$$\left(e^{u/2} - e^{-u/2}\frac{d\Psi_0(e^u)}{du} - \frac{e^{u/2}}{e^{3u}-e^u}\right)\sqrt{\frac{u}{\pi}}e^{-i\frac{\pi}{4}} = Z_u(0) \qquad (1.5)$$

Expression (1.8) tells us that proving RH is equivalent to show that the ODE given in (1.5) with $\{d_k\} \in R$ and $d_k = \frac{1}{k!}\frac{d^kV(x)}{dx^k}\Big|_{x=0}$, $V(x) = \sum_{k=0}^{\infty} d_k x^k$ using (1.5) together with a finite power expansion for V(x) , using (1.5) we could obtain the constants $\{d_k\} \in R$ to get an approximate solution for the potential V(x).

If RH is true and $\zeta\left(\frac{1}{2}+iE_n\right)=0$, with $E_n = -E_{-n}$ being the eigenvalues of a certain operator $H = p^2 + V(x)$, using expression (1.2) and the functional equation $\zeta(1-s) = 2(2\pi)^{-s} Cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$,then for $n \ge 0$ we can define an spectral Zeta function , involving the nontrivial zeros of Zeta and primes and prime powers

$$\sum_{n=0}^{\infty}\frac{1}{E_n^{\ s}} = \frac{Sec\left(\frac{\pi s}{2}\right)}{2\Gamma(s)}\int_0^{\infty} dt Tr\left\{e^{it\hat{H}}\right\} = \left(2\pi\right)^{-s}\frac{\zeta(s)}{\zeta(1-s)}\int_0^{\infty} dt e^{t/2}\left(1-e^{-t}\frac{d\Psi_0(e^t)}{dx}-\frac{1}{e^{3t}-e^t}\right)t^{s-1} \tag{1.6}$$

The value $\prod_{n=0}^{\infty} E_n = e^{-\frac{d\zeta(0)}{ds}}$ would be the regularized product of all the positive 'Eigenvalues' $\{E_n\}$ this expression can also be used to obtain a Dirac measure for the E_n , let us introduce

$$\sum_{n=0}^{\infty} E_n^{\ -s} = \int_0^{\infty} dt\left(\sum_{n=0}^{\infty}\frac{1}{E_n}\delta\left(t-\frac{1}{E_n}\right)\right)t^{s-1} \qquad \frac{\zeta(s)}{\zeta(1-s)} = \int_0^{\infty} dt K_0(t) t^{s-1} \tag{1.7}$$

Using the properties of the Mellin transform applied to solve linear integral operators $I[f] = \int_0^{\infty} dt R(xt) f(t)$, if we combine (1.6) and (1.7) we get the result

$$\Theta(x) = \sum_{n=0}^{\infty}\frac{1}{E_n}\delta\left(x-\frac{1}{E_n}\right) = \int_0^{\infty}\frac{dt}{t}K_0(2\pi xt)\left(e^{1/2t}+\frac{e^{-1/2t}}{t^2}\frac{d\Psi_0(e^{1/t})}{dt}-\frac{e^{1/2t}}{e^{3/t}-e^{1/t}}\right) \tag{1.8}$$

If we took the Mellin transform $\int_0^{\infty} dx x^{s-1}$ inside (1.8) together with the change of variable xt=z we would recover equation (1.6) , note that the Mellin transform of the Kernel $K_0(2\pi xt)$ does not depend on the nontrivial zeros $\rho = \frac{1}{2}+it$.

Using test functions $\frac{1}{x}h\left(\frac{1}{2}\pm\frac{i}{x}\right)$ inside (1.8) obtained from our Trace formula for $Tr\left\{e^{iu\hat{H}}\right\}$ we can relate the convergent sum $\sum_{\rho} h(\rho)$ to a sum over primes and prime powers

$$\int_0^1 dxh\left(\frac{1}{2}+\frac{i}{x}\right)\int_0^\infty dt\frac{K_0(2\pi xt)}{xt}\left(e^{1/2t}+\frac{e^{-1/2t}}{t^2}\frac{d\Psi_0(e^{1/t})}{dt}-\frac{e^{1/2t}}{e^{3/t}-e^{1/t}}\right)+c.c=\sum_\rho h(\rho) \tag{1.9}$$

Formula (1.9) and its result can be compared with sums $\sum_\rho \frac{a^\rho}{\rho}$ (explicit formula for Chebyshev function) and $Z(n)=\sum_\rho \frac{1}{\rho^n}$ $n\in N$, that can be calculated exactly.

- *The Trace* $Tr\left\{e^{iu\hat{H}}\right\}$ *and the sum* $\sum_\gamma h(\gamma)$

Even though we can not solve equation (1.3) we can use the Trace expression (1.2) to find stimates for sums $\sum_\gamma h(\gamma)$. First we define a couple of function g(x) and h(x) with the following properties

- Both g(x)=g(-x) and h(x)=h(-x) are even functions
- $\lim_{x\to 0}\frac{g(x)}{x}$ exists and it is finite
- The functions h(x) and g(x) are related by a Fourier Cosine transorm $\frac{1}{\pi}\int_0^\infty dxh(x)Cos(\alpha x)=g(\alpha)$
- The function h(x) can be defined by analytic continuation to the region of complex plane defined by Re (s) =0 , in particular $h(\pm i/2)$ $\sqrt{-1}=i$

If RH (Riemann Hypothesis) is true, then the Trace (1.3) is just a sum of cosines $\sum_{\gamma>0} 2\cos(\gamma u)$, then if we take g(u) as a test function

$$\int_0^\infty dug(u)Tr\left\{e^{iu\hat{H}}\right\}=\frac{h(i/2)+h(-i/2)}{2}-\sum_{n=1}^\infty\frac{\Lambda(n)}{\sqrt{n}}g(\log n)-\int_0^\infty due^{-u/2}\frac{g(u)}{e^{2u}-1}=2\sum_{\gamma>0}h(\gamma) \tag{1.10}$$

In order to obtain (1.10) we have used the representation in terms of Dirac deltas of the derivative of Chebyshev function to get $\int_0^\infty dug(u)e^{-u/2}\frac{d\Psi_0(e^u)}{du}=\sum_{n=1}^\infty\frac{\Lambda(n)}{\sqrt{n}}g(\log n)$, and the Euler formula for cosine to represent the integral $\int_0^\infty dug(u)e^{u/2}$ as the sum $\frac{1}{2}\left(h(i/2)+h(-i/2)\right)$. An special case is whenever we choose

$h(x)=\frac{\pi}{2}\left(\delta(s-x)+\delta(s+x)\right)$ and $g(u)=\cos(u)$ (1.11)

Then we can use the functions in (1.11) and the formula (1.10) to get

$$\sum_{\gamma}\pi\delta(s-\gamma)=\frac{1}{2}\left(\frac{\zeta'(1/2+is)}{\zeta(1/2+is)}+\frac{\zeta'(1/2-is)}{\zeta(1/2-is)}\right)-\frac{2}{1+4s^{2}}-\sum_{n=0}^{\infty}\frac{(2n+1/2)}{\left((2n+1/2)^{2}+s^{2}\right)} \tag{1.12}$$

(1.12) is the 'density of states' in QM , and can be used to know how many zeros of the form ½+is are with imaginary part less that a given 'T' since $N(T)=\int_{0}^{T}ds\left(\sum_{\gamma>0}\delta(s-\gamma)\right)$

A formal derivation of (1.12) can be obtained considering the following indentities for divergent series involving zeta regularization or analytic continuation

$$\sum_{n=0}^{\infty}e^{an}=\frac{1}{1-e^{a}}\quad \text{(linearity)}\qquad \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2-is}}=-\frac{\zeta'(1/2-is)}{\zeta(1/2-is)} \tag{1.13}$$

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\cos(s\log n)=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}n^{is}+\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}n^{-is}\right) \tag{1.14}$$

And the Laplace transform of Cosine $\int_{0}^{\infty}dte^{-st}\cos(at)=s(s^{2}+a^{2})^{-1}$ .The poles inside (1.14) are of three kinds $s=\gamma_{n}$ from the Non-trivial zeros of Zeta function , $s=\pm i/2$ due to the divergent value $\zeta(1)$ and $s=\pm i\left(1/2+2n\right)$ $\forall n\in N$ form the trivial zeros of zeta function -2,-4,-6,..........

- *Riemann-Weyl formula and a solution for the inverse of potential V(x):*

A similar formula to (1.10) had been previously introduced by Weyl in 1972 [9]

$$\sum_{\gamma}h(\gamma)=h(i/2)+h(-i/2)-g(0)\log\pi-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)dr \tag{1.15}$$

Weyl summation formula can be used to solve equation (1.10) if we make inside this integral equation the change of variable $x=V^{-1}(\xi)$, then (1.10) is simply proportional to the inverse Fourier transform of $\sum_{\gamma}\frac{e^{-iu\xi+i\pi/4+iu\gamma}}{\sqrt{u}}$ on the interval $(0,\infty)$ which is just proportional to another sum $\sum_{\gamma}\frac{1}{\sqrt{|\xi-\gamma|}}$ involving the imaginary parts of the Riemann zeta zeros , to get rid off the sum we ca use (1.15) to express the inverse of potential

$$V^{-1}(\xi) \approx \frac{A}{\sqrt{|2\xi+i|}} + BCos(c\xi+\pi/4) + D +$$

$$E\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos(\xi\log n+\pi/4)}{|n\log n|^{1/2}} + Fp.v\left(\int_{-\infty}^{\infty} dr \frac{\Gamma}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)\left(\frac{1}{|\xi+r|^{1/2}}+\frac{1}{|\xi-r|^{1/2}}\right)\right) \tag{1.16}$$

Where $A,B,c,D,E,F \in R$ are real parameters that define the potential , from formula (1.16) the Hamiltonian would be self adjoint and its energies (imaginary part of zeros) would be real numbers. (If the new limits of integration after taking the change of variable $x = V^{-1}(r)$ were not $\pm\infty$ but real numbers c,d then we could multiply the left side of equation by $W_c^d(x) = H(x-c) - H(x-d)$, $c,d = V(\pm\infty)$)

The sum involving $\Lambda(n)$ can be treated using fractional calculus and zeta regularization

$$2\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos(\xi\log n+\pi/4)}{|n\log n|^{1/2}} \to -\sqrt{-i}\,\frac{d^{-1/2}}{d\xi^{-1/2}}\left(\frac{\zeta'(1/2+i\xi)}{\zeta(1/2+i\xi)}\right) - \sqrt{i}\,\frac{d^{-1/2}}{d\xi^{-1/2}}\left(\frac{\zeta'(1/2-i\xi)}{\zeta(1/2-i\xi)}\right) \tag{1.17}$$

At the points $\xi = \gamma$ the inverse of potential becomes ∞ , as we can expect from

$$\sum_{\gamma}|\xi-\gamma|^{-1/2} = \sum_{\gamma>0}|\xi-\gamma|^{-1/2} + \sum_{\gamma>0}|\xi+\gamma|^{-1/2} \qquad (1.18)$$

Although we have investigated the trace involving the Chebyshev function $\Psi_0(x) = \sum_{n<x}\Lambda(n)$, with some changes it can also be applied to find a Trace involving the Mertens function $M_0(x) = \sum_{n<x}\mu(n) = \begin{cases} M(x) - \frac{1}{2}\mu(x) & x \in Z^+ \\ M(x) & otherwise \end{cases}$, if there are no multiple zeros of Riemann zeta function so $\zeta'(\rho) \neq 0$, then we can find an expression for Mertens function involving a sum over Zeta zeros as

$$M_0(x) = \sum_{\rho}\frac{x^{\rho}}{\rho\zeta'(\rho)} - 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!n\zeta(2n+1)}\left(\frac{2\pi}{x}\right)^{2n} \quad x>0 \quad (1.19)$$

Since the Mertens function is just an step function its first derivative it will be a set of Dirac delta function so

$$\int_0^{\infty} due^{-u/2}\,\frac{dM_0(e^u)}{du}g(u) = \sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}\,g(\log n) \quad (1.20)$$

Where $\mu(n) = \begin{cases} 0 \text{ if n is not square-free.} \\ 1 \quad n=1 \\ (-1)^k \text{ if n is square-free with k- distinct prime factor} \end{cases}$ is Mobius function

Then differentiating respect to ‘x’ and setting $x = e^{u}$, assuming RH is true so all the non-trivial zeros are of the form $\rho = 1/2 + it$, (with ‘t’ real) and choosing two even test functions (g, h) related by a Fourier transform $g(x) = \frac{1}{\pi}\int_0^{\infty} h(u)\cos(ux)du$ one gets

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}g(\log n) = \sum_{\gamma}\frac{h(\gamma)}{\zeta'\left(\frac{1}{2}+i\gamma\right)} + \sum_{n=1}^{\infty}\frac{2(2\pi)^{2n}(-1)^{n}}{(2n)!\zeta(2n+1)}\int_{-\infty}^{\infty} dug(u)e^{-(2n+1/2)u} \qquad (1.21)$$

(1.21) is a similar expression to Riemann-Weyl explicit formula for the sum relating primes and Riemann Zeta zeros. As an example let be $g(x) = \begin{cases} e^{-\varepsilon x} & \text{when x > 0} \\ 0 & \text{when x < 0} \end{cases}$ with $'\varepsilon'$ a real and positive number, if $\varepsilon = 1/2$ one should have the Prime Number theorem , so $\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0$, an even stronger conclusion is that if RH is true then $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1/2+\varepsilon}}$ must be convergent for every positive $\varepsilon \to 0$ and equal to

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1/2+\varepsilon}} = \sum_{\gamma}\frac{1}{\zeta'(1/2+i\gamma)}.\frac{1}{\gamma+i\varepsilon} + \sum_{n=1}^{\infty}\frac{4(2\pi)^{2n}(-1)^{n}}{(2n)!\zeta(2n+1)}\frac{1}{4n+1+2\varepsilon} \qquad (1.22)$$

2. Zeta regularization for divergent integrals:

Given the function $f(x) = x^{m}$, we can use the Euler-Maclaurin summation formula to obtain a recurrence relation between an integral of the form $I(m,\Lambda) = \int_0^{\Lambda} p^{m}dp \quad m \in \mathbf{Z}^{+}$ with $m\int_0^{\Lambda} x^{m-1}dx = \Lambda^{m}$ and the series $\sum_{i=0}^{\Lambda-1} i^{m}$, $m \geq 0$ ref [7]

$$I(m,\Lambda) = (m/2)I(m-1,\Lambda) + \sum_{i=0}^{\Lambda-1} i^{m} - \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}(m-2r+1)I(m-2r,\Lambda)$$

(2.1)

$$\int_0^{\infty} x^{m}dx = \frac{m}{2}\int_0^{\infty} x^{m-1}dx + \zeta(-m) - \sum_{r=1}^{\infty}\frac{B_{2r}m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^{\infty} x^{m-2r}dx \qquad \Lambda \to \infty$$

The coefficients $a_{mr} = \frac{\Gamma(m+1)}{\Gamma(m-2r+2)}$ vanish when $m+2 \leq 2r$, hence the sum inside (2.1) is finite if m is an integer , in the physical limit the cutoff $\Lambda \to \infty$, this makes the series $\sum_{i=0}^{\Lambda-1} i^m$ to be divergent for $m \geq -1$, in this case we should use the Functional equation for the Zeta function to obtain the (Regularized) value

$$\lim_{\Lambda\to\infty}\sum_{n=1}^{\Lambda-1} n^m = 1+2^m+3^m+\ldots+\Lambda^m \to \zeta_R(-m) = \zeta(-m) \tag{2.2}$$

(2.2) is the Zeta-regularized value for the divergent sum envolved in (2.1) , using this method we can compute the divergent integrals $I(m,\Lambda) \quad \Lambda\to\infty$, for m=1,2,3

$$I(0,\Lambda) = \zeta(0) = -1/2 = \int_0^\infty dx$$

$$I(1,\Lambda) = \frac{I(0,\Lambda)}{2} + \zeta(-1) = \int_0^\infty x dx$$

$$I(2,\Lambda) = \left(I(0,\Lambda)\frac{1}{2} + \zeta(-1) \right) - \frac{B_2}{2} a_{21} I(0,\Lambda) = \int_0^\infty x^2 dx \tag{2.3}$$

$$I(3,\Lambda) = \frac{3}{2}\left(\frac{1}{2}\left(I(0,\Lambda)+\zeta(-1)\right) - \frac{B_2}{2} a_{21} I(0,\Lambda) \right) + \zeta(-3) - B_2 a_{31} I(0,\Lambda) = \int_0^\infty x^3 dx$$

The case m=0 is just equal to the divergent series 1+1+1+1+1+1+1+1+1+1+... taking the regularized value -1/2 evaluated from $\zeta(0)$

For an arbitrary function f(x) so its integral would diverge as a power of the cutoff Λ^{N+1} we could expand f(x) into a Laurent series convergent for |x| <1 and |x| >1 so we find

$$\int_a^\Lambda dx f(x) = \sum_{r=0}^{N} c_r I(r,\Lambda) + c_{-1} I(a,-1,\Lambda) + O(\Lambda^{-1}) - \sum_{i=0}^{N}\int_0^a dx\left(c_i x^i\right) - \sum_{j=2}^{\infty} c_{-j} a^{-j+1} \tag{2.4}$$

$\{c_i\} \in \mathbb{R}$, taking $\Lambda\to\infty$, and using (2.1) (2.2) (2.3) to regularize the divergent integrals $I(m,\Lambda)$ we could obtain a regularized (finite) value for the integral $\int_0^\Lambda dx f(x)$, however the logarithmic divergent integral $I(a,-1,\Lambda) = \int_a^\Lambda \frac{dx}{x}$ can not regularized by our formulae, the solution would be to use the Euler-Maclaurin summation to approximate the divergent integral by a divergent Harmonic sum that can be attached a 'Ramanujan sum' $\gamma - \sum_{n=1}^{a} \frac{1}{n}$ (γ =Euler-Mascheroni constant)

- *Zeta regularized product of distributions:*

Formulae (2.1-2.3) can be used to compute divergent integrals of the form $\int_0^\infty x^{s-1}dx$, but also could give an answer to the problem of multiplication of two distributions involving Dirac delta and its derivatives $D^m\delta(x)$, if we tried to define the product of distributions involving delta functions we could use the 'convolution theorem' applied to the Fourier transform (A=normalization constant) :

$$(2\pi)^2 i^{m+n} D^m\delta(\omega)D^n\delta(\omega) = F_\omega\left(x^m * x^n\right) = AF_\omega\left\{\int_{-\infty}^{\infty} dt t^m (x-t)^n\right\} \qquad (2.5)$$

Unfortunately (2.5) makes no sense , the integral is divergent for every real or complex value of 'x' , if m and n are positive integers using the Binomial expansion

$$i^{m+n} D^m\delta(\omega)D^n\delta(\omega) = \sum_{k=0}^{n}\binom{n}{k} i^{m+k} AD^{n-k}\delta(\omega)(-1)^k i^{n-k} D^{m+k}\delta(0)^{[R]} \qquad (2.6)$$

$$i^{m+n} D^m\delta(\omega)D^n\delta(\omega) = \sum_{k=0}^{n}\binom{n}{k} AD^{n-k}\delta(\omega)(-1)^k i^{n-k}\left((-1)^{m+k}+1\right)\int_0^\infty x^{m+k}dx \qquad (2.7)$$

'R' stands for regularization (regularized value) , the divergent integrals come now from the dirac delta and its derivatives evaluated at x=0 , which are proportional to $\int_{-\infty}^{\infty} x^k dx$ for k=2r+1 (Odd) the integral considered in Principal Value is 0 , for k=2r (even integer) the integral can be written as $i^{2r}D^{2r}\delta(0) = 2I(2r,\Lambda) \quad \Lambda\to\infty$, $I(2,r) = \int_0^\Lambda x^{2r}dx$ (r=integer) and can be evaluated using (2.1) and (2.2) .

The expression (2.7) is real ,this is what one would expect since the product of two distributions taking only real values must be real , however (2.6) is not still invariant under the change $m \to n$ and $n \to m$ (this is a mistake we made in paper [7]) so we should take a more symmetrical product of distributions defined by

$$\left(D^m\delta \otimes D^n\delta\right)_R(\omega) = \frac{1}{2}\left(D^m\delta(\omega)D^n\delta(\omega) + D^n\delta(\omega)D^m\delta(\omega)\right) \qquad (2.8)$$

The simplest case is m=n=0 so $\left(\delta\otimes\delta\right)_R(\omega) = -A\delta(\omega)$

For the case of 'm' and 'n' not being an integer or we have a shifted dirac delta $D^k\delta(x-a)$, we could use the identies for the k-th power of 'x' or the traslation operator e^D and $D=\frac{d}{dx}$ in the form

$$e^{-aD}D^r\delta(x)=\sum_{j=0}^{\infty}(-1)^j\frac{a^j}{j!}D^{r+j}\delta(x)=\delta(x-a) \qquad D^r=\sum_{k=0}^{\infty}\binom{r}{k}(D-1)^k \qquad (2.9)$$

In case of integrals on R^d $\int_{R^d} dkF(\vec{k})$, if the function F , is invariant under Lorentz transformations, then making a Wick rotation to imaginary time $t\to it$,the metric becomes $ds^2=dx^2+dy^2+dz^2+dt^2$ which is invariant under rotations, taking 4-dimensional polar coordinates our integral can be evaluated as $\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^{\infty} drf(r)r^{d-1}$, if not we could replace the integral over the cross section (angles) $d\Omega$ by a discrete sum $\sum_i\int_0^{\infty} drf(r,\Omega_i)r^{d-1}$, with 'd' equal to the dimension of space-time

- **Example:** $\int_a^{\Lambda} dx\frac{x^2}{1+x}$ with $\Lambda\to\infty$ in this case the integral has a power-law (quadratic) divergence Λ^2 , a >1 and integer (this is not relevant since the integral diverges only for big 'x') , the Laurent series for |x| bigger than 1 is $x-1+x^{-1}+\sum_{j=3}^{\infty}(-1)^j x^{1-j}$, if we approximate the logarithmic divergent integral of 1/x by the divergent series $\sum_{n=0}^{\infty}\frac{1}{n+a}$ (after a change of variable x=t+a) then, the approximate 'Zeta regularized' value of the integral would be

$$\left(\int_a^{\infty} dx\frac{x^2}{1+x}\right)_R=\frac{-\zeta(0)}{2}+\zeta(-1)-\frac{\Gamma'(a)}{\Gamma(a)}+a-\frac{a^2}{2}+\sum_{j=3}^{\infty}\frac{(-1)^j}{j-2}a^{2-j} \qquad (2.10)$$

Another example without a logarithmic divergence , would be $\int_a^{\Lambda} dx\frac{x^4}{(1+x^2)}$ in this case the regularized finite value is just $\zeta(0)-a+\sum_{n=2}^{\infty}\frac{n(-1)^n}{2n-3}a^{3-2n}$, the logarithmic derivative of Gamma function inside (2.10) is just the Ramanujan resummation of the Hurwitz series $\zeta_H(1,a)$ avoiding the pole at s=1

Another method to evaluate these kind of divergent integrals is, substract a sum of the form $\sum_{i=-1}^{N}c_i x^i$, so the integral defined by $\int_0^{\infty} dx\left(f(x)-\sum_{i=-1}^{N}c_i x^i\right)dx$ exists in the Riemann sense , in case we had a Fourier transform, we would add and substract terms in the form $\sum_{i=-1}^{N}c_i x^i e^{iux}$ which are proportional to the derivatives of $\delta(u)$.

Example $\int_3^\infty \frac{x}{x+2}dx = \int_3^\infty \frac{4}{x(x+2)}dx + \int_3^\infty 1dx - \int_3^\infty \frac{2}{x}dx$ the first integral on the right of = is convergent , the logarithmic integral can be (approximately) regularized by means of Ramanujan sumation to the finite value $2\frac{\Gamma'(3)}{\Gamma(3)}$ and the other integral is just the regulariez value $\zeta(0)-3$.

On R^k simply use the k-dimensional polar coordinates and try using substraction to obtain an integral of the form $\int_0^\infty dr\left(f(r,\Omega)-\sum_{i=-1}^{N} r^i U_i(\Omega)\right)$, the idea here is isolate the divergent integrals of the form $\int_0^\infty r^m dr$

Conclusions and final remark

In this paper we have used the method of Zeta regularization of series applied to the problem of finding finite results for divergent integrals $\int_0^\infty x^m dx$ and to give an adequate Hilbert-Polya operator in order to solve Riemann Hypothesis.

On the first part of the paper we show that if RH is true then the Chebyshev function evaluated at x=exp(u) is just the trace of the exponential of a Hamiltonian $H = p^2 + V(x)$ whose eigenvalues are precisely the imaginary part of the nontrivial zeros, we extend this idea and define the distribution $Z(u) = \sum_{m=-\infty}^{\infty} e^{iu\gamma_m} = 2\sum_{m=0}^{\infty}\cos(\gamma_m u)$ which can be calculated for every 'u' bigger than 0 and whose value is related to the derivative of Chebyshev function $\frac{d\Psi_0(e^u)}{du}$. We also discuss the applications of this trace formula for $Z(u) = \sum_{m=-\infty}^{\infty} e^{iu\gamma_m}$ and how can be used to obtain the values of the sum $\sum_\gamma f(\gamma)$ in a similar way to Riemann-Weyl explicit formula, we also obtain a method to calculate $\sum_{\mathrm{Im}\{\rho\}<T} 1 = N(T)$, using zeta regularization we obtain an exact expression for the oscillating term in N(T) as $\frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+iT\right)\right)$ which comes from integratin the regularized value of the series $\sum_{n=1}^{\infty}\Lambda(n)\frac{\cos(s\log n)}{\sqrt{n}}$

It may seem at first sight that there is no relation between Rieman-Weyl formula for the sum $\sum_{\gamma>0} f(\gamma)$ and the one we obtained in (1.10) , however we can prove the following identity (in the sense of distributions)

$$\int_0^\infty du\left(e^{u/2}+\frac{e^{-u/2}}{1-e^{2u}}\right)\frac{\sin(ux)}{2u}-\frac{1}{2}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}\log n}\sin(x\log n)=$$
$$\mathrm{Im}\left\{\log\zeta\left(\frac{1}{2}+ix\right)+\log\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right\}-\frac{1}{2}x\log\pi+\pi \qquad (2.11)$$

This formula can be inmediatly obtained from the definition of the function that gives us the number of zeros with imaginary part less than a given 'x' N(x) ,the property of the Chebyshev function $\Psi_0(e^u)$ and the properties of Fourier inverse sine and cosine transforms for our Trace $Z(u)=2\sum_{\gamma>0}\cos(u\gamma)$ with $Z(u)=2u\int_0^\infty dxN(x)\sin(ux)$ and

$$N(x)=\frac{1}{\pi}\mathrm{Im}\left\{\log\zeta\left(\frac{1}{2}+ix\right)+\log\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)\right\}-\frac{1}{2\pi}x\log\pi+1 \qquad (2.12)$$

Taking the inverse sine transform and the definition of Z(u) for u >0 we recover the above formula relating a Fourier sine inverse transform and the imaginary parts of the logarithm of $\zeta\left(\frac{1}{2}+ix\right)$ and $\Gamma\left(\frac{1}{4}+\frac{ix}{2}\right)$, if we differentiate respect to 'x' and use even test functions g(x) and h(x) with $\frac{1}{2\pi}\int_{-\infty}^{\infty}drh(r)e^{irx}=g(x)$,and taking into account that we can expand the Real part of $\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)$ into the divergent (but regularized) sum $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\cos(s\log n)$, then the oscillating part of the N(x) gives us the term proportional to $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)$, integrating over 'x' on (2.11) the right terms on the derivative of (2.11) is precisely the Riemann-Weyl contribution to the sum $\sum_{\gamma>0}h(\gamma)$ $\int_{-\infty}^{\infty}\frac{dx}{2\pi}h(x)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)-g(0)\log\pi$ the term on the left would be $\int_0^\infty du\frac{g(u)e^{-u/2}}{e^{2u}-1}$, which is just the expression for the sum $\sum_{\gamma>0}h(\gamma)$ obtained from our trace Z(u) , remember that for u <0 from the definition of Chebyshev functions in terms of the explicit formula, there is a factor '2' since the Fourier cosine and Fourier exponential transforms for even functions are related by $2F_c\{g(x)\}=F_e\{g(x)\}$

On the second part of the paper, we use the Euler-Maclaurin sum formula together with the analytic continuation of the Zeta function $\zeta(s)$ to negative values , note that E-M (Euler Maclaurin) formula is correct for integrals of the form $\int_1^\infty\frac{dx}{x^\alpha}$ whenever $\alpha\geq1$

since the series $\sum_{n=2}^{\infty} n^{-\alpha} = \zeta(\alpha) - 1$ is convergent, the idea of our method is to use Euler-Maclaurin formula and then perform an analytic continuation to values with $\alpha < 1$, in order to calculate the integral $\int_0^{\infty} x^{\alpha} dx \quad \alpha > 0$, so $\sum_{n=0}^{\infty} n^{\alpha} = \zeta(-\alpha)$ and we can define a recurrence equation for every integral such as formula (2.1) being the initial term in the recurrence $\int_0^{\infty} dx = \sum_{n=0}^{\infty} 1 = \zeta(0)$, this kind of zeta regularization of series would allow to calculate divergent integrals and to define a regularized product of distributions $D^m \delta(\omega) D^n \delta(\omega)$ by applying Convolution theorem to the weird function $\int_{-\infty}^{\infty} dt t^m (x-t)^n \quad (m,n) \in Z$ which can not be defined for any 'x' unless we know how to calculate divergent integrals. Although we can not use (1.7) in order to regularize $\int_0^{\infty} \frac{dx}{x+a}$ we can approximate this divergent integral by the Hurwitz series $\zeta_H(1,a)$ which is still divergent but can be assigned a finite value in the sense of 'Ramanujan resummation' $-\frac{\Gamma'(a)}{\Gamma(a)}$ another alternative would be differentiating respecto to 'a' to get a convergent integral so we have the result $\log(a) + C$ with 'C' an infinite constant.

Appendix A: an integral Trace for the Green function

A formula for the sum $\sum_{n=0}^{\infty} \delta(E - E_n)$ in terms of the Trace of the 'Resolvent' (green function) of a Quantum Hamiltonian $\hat{H}\phi_n = E_n \phi_n$ can be defined as:

$$Tr\{G(x,x',E)\} = \int_{R^4} d^4 x G(x,x,E) = \pi \sum_{n=0}^{\infty} \delta(E - E_n) \qquad G(x,x',E) = \frac{1}{E + i\varepsilon - \hat{H}} \quad \text{(A.1)}$$

One of the easiest method to prove this , is to consider that given a convergent series with sum S and its Borel transform B(s) defined by $B(S, a_n) = \int_0^{\infty} dt \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \right) e^{-t}$ then

S=B(S) , $S = \sum_{n=0}^{\infty} a_n$ in this case if we take the series

$$\frac{1}{E + i\varepsilon - H} = \frac{E^{-1}}{1 + \frac{(i\varepsilon - H)}{E}} = E^{-1} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (i\varepsilon - H)^n x^n}{E^n n!} \right) = \int_0^{\infty} dt e^{-t(1 + i\varepsilon - \hat{H})} \quad \text{(A.2)}$$

Where ε is an small number so $\varepsilon \to 0$, then using the formula for the Principal value

$P.V\left(\frac{1}{x}\right)=i\pi\delta(x)+\frac{1}{x+i\varepsilon}$, in this case taking the trace of the operator inside (A.2) we can give a proof to (A.1) using the technique of Borel resummation.

Another example of the method of Borel resummation , let be $P(x)=\sum_{n=0}^{\infty}(-1)^n\alpha(n)x^n$ the generating function of the coefficients $\alpha(n)$, let be the function f(t) defined by $\alpha(s-1)=\int_0^{\infty}dtf(t)t^{s-1}$ then using again the Borel-generalized resummation

$$P(x)=\int_0^{\infty}dt\left(\sum_{n=0}^{\infty}(-1)^n(xt)^n\right)f(t)=\sum_{n=0}^{\infty}(-1)^n\alpha(n)x^n \quad \text{or} \quad P(x)=\int_0^{\infty}dt\frac{f(t)}{1+xt} \qquad \text{(A.3)}$$

If we took the Mellin transform on both sides $\int_0^{\infty}dxx^{s-1}$ one would find $\hat{P}(s)=\hat{K}(s)\hat{F}(1-s)$, or in terms of improper integrals

$$\int_0^{\infty}dt\left(\sum_{n=0}^{\infty}\alpha(n)(-x)^n\right)=\frac{\pi\alpha(-s)}{\sin(\pi s)} \qquad \text{since} \quad \int_0^{\infty}dtf(t)t^{-s}=\alpha(-s) \qquad \text{(A.5)}$$

This last formula is known as 'Ramanujan Master theorem' , note that we have proved this only using the fact that for a convergent series its sums and Borel transform must be equal S=B(S). According to tihs formula our K function defined previously on (1.7) would be equal to $K_0(x)=-\sum_{n=0}^{\infty}\frac{\sin(n\pi)}{\pi}.\frac{\zeta(-n)}{\zeta(n+1)}(-x)^n$

Appendix B: A Riemann-Weyl summation formula

Riemann-Weyl formula, is a good tool to calculate sums over the imaginary parts of the Riemann zeros $\sum_{\gamma}f(\gamma)$, using Zeta regularization and the main property of Dirac delta distribution $\int_{-\infty}^{\infty}dx\delta(x-a)g(x)=g(a)$ together with the Hadamard product representation $\zeta(z)=\frac{2\pi^{z/2}}{(z-1)z}.\prod_{\rho}\left(1-\frac{z}{\rho}\right).\frac{1}{\Gamma(z/2)}$ we can give a proof of it , we will also need the following formulae

$$\lim_{\varepsilon\to 0}\frac{1}{x+i\varepsilon}=-i\pi\delta(x)+P\left(\frac{1}{x}\right) \quad \text{and} \qquad -\frac{\zeta'(1/2+i\varepsilon+is)}{\zeta(1/2+i\varepsilon+is)}=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2+i\varepsilon}}.e^{is\log(n)} \qquad \text{(B.1)}$$

The first is Sokhotsky's formula for delta distribution and the second is just the usual zeta regularization procedure for the Dirichlet sum associated to Mangoldt function , taking logarithms inside the Hadamard product and differntiating at the point $z=\frac{1}{2}+is+\varepsilon$, where epsilon is an small quantity so we can ignore cuadratic terms $\varepsilon^2=0$, we can now obtain the following formula

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)=\frac{1}{2}\log(\pi)+\sum_{\gamma}\pi\delta(s-\gamma)-\frac{1+2is}{\frac{1}{4}+s^2}-\frac{1}{2}.\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{s}{2}+\frac{i\varepsilon}{2}\right)+iP.V\left(\sum_{\gamma}\frac{1}{s-\gamma}\right)$$

(B.2)

To obtain (B.2) we have used both formulae in (B.1) so $\sum_{\gamma}\pi\delta(s-\gamma)=\sum_{\gamma}\frac{i}{s-\gamma+i\varepsilon}+\sum_{\gamma}iP\left(\frac{1}{s-\gamma}\right)$ with $\varepsilon\to 0$ and the derivative of $\log\left(\frac{1}{2}+is+i\varepsilon\right)$ can be calculated using Zeta regularization in terms of a sum involving the Mangoldt function $\Lambda(n)$ so (B.3) reads now

$$\sum_{\gamma}\pi\delta(s-\gamma)=\frac{1+2is}{\frac{1}{4}+s^2}+\frac{1}{2}.\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{s}{2}+\frac{i\varepsilon}{2}\right)-\frac{1}{2}\log(\pi)-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2+i\varepsilon}}.e^{is\log(n)}+iP.V\left(\sum_{\gamma}\frac{1}{s-\gamma}\right)$$

(B.3)

If we use inside (B.3) the test functions g(x) and h(x) related by a Fourier transform so $g(u)=\int_{-\infty}^{\infty}\frac{dx}{2\pi}h(x)e^{iux}$ with g(x) and h(x) being even functions satisfying certain properties, in Cauchy's principal value $\int_{-\infty}^{\infty}\frac{dx}{2}h(x)=\int_{-\infty}^{\infty}dx\frac{h(x)\sin(bx)}{\frac{1}{4}+x^2}=0=\int_{-\infty}^{\infty}dx\frac{h(x)x}{\frac{1}{4}+x^2}$

and if h(x) can be continued analytically to the critical strip we find

$$\frac{1}{\pi}.\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}d\alpha\frac{g(\alpha)}{\frac{1}{4}+x^2}e^{-i\alpha x}=\frac{1}{\pi}.(2\pi)\int_{-\infty}^{\infty}d\alpha g(\alpha)e^{-\alpha/2}=\frac{1}{2}.\left(h(i/2)+h(-i/2)\right).2 \qquad \text{(B.4)}$$

In order to change the order of integration inside (B.4) , we assume h(x) and g(x) are regular enough and apply Fubini's theorem .

Appendix C: Generalization of Urysohn Non-linear equation

The problem with integral equation (1.3) is the fact that this implies solving an integral equation with a distribution $\frac{d\Psi_0(e^u)}{du}$, ths kind of integral equation can be generalized to include test functions (g,h) with the following properties

- $g(r,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du h(r,u) e^{-iux}$ for a real parameter 'r' with g and h being even functions on the variable 'x' $g(r,x) = g(r,-x)$
- $\frac{g(r,x)}{x}$ is finite in the limit $x \to 0$
- The integral of $g(r,x)e^{x/2+iur}$ on $x \in (-\infty,\infty)$, and the sum $\sum_{n=1}^{\infty} \frac{\Lambda(n) g(r,\log n)}{\sqrt{\frac{n}{\log n}}}$
- $\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dr |h(r,u)|^2 < \infty$, and the same holds for g(r,x) so the nonlinear Kernel are on L^2
- $g(r,x)$ and $h(r,x)$ are regular enough so we can use Fubini's theorem to interchange the order of integration

Then introducing this tests functions on (1.3) and taking the integral over 'u' interchanging the order of integration and using $h(r,x) = \int_{-\infty}^{\infty} du g(r,u) e^{iuV(x)+i\pi/4}$ we have the Urysohn integral equation (depending on the choose of g)

$$\int_{0}^{\infty} du \sqrt{u} g(r,u) \left(e^{u/2} + \frac{e^{-u/2}}{1-e^{2u}} \right) - \sum_{n=1}^{\infty} \frac{\Lambda(n) g(r,\log n)}{\sqrt{\frac{n}{\log n}}} = \sqrt{\pi} \int_{-\infty}^{\infty} dx h\left(r, V(x) + \frac{\pi}{4} \right) \quad \text{(C.1)}$$

(C.1) is the generalization of (1.3) with the advantage that we are dealing with functions instead of distributions , V(x) depends on Mangoldt function $\Lambda(n)$ this is not casual since the primes and Riemann Zeta zeros are related . We have used the semiclassical approximation for the series (in the sense of distribution) $Z(u) = \sum_{m=-\infty}^{\infty} e^{iuE_n}$, $E_n \in R$ (if RH correct) , this Z(u) will depend on the derivative of Chebyshev function $\frac{d\Psi_0(e^u)}{du}$

In a similar manner we can use the properties of the Laplace transform of $\frac{\sin(at)}{t}$ and use the analytic continuation to express (2.11)

$$\begin{aligned} &\tan^{-1}(-2x) + \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{2x}{1-4n} \right) + \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}\log n} \sin(x\log n) = 2\pi + \\ &2\,\mathrm{Im}\left\{ \log\zeta\left(\frac{1}{2} + ix \right) + \log\Gamma\left(\frac{1}{4} + \frac{ix}{2} \right) \right\} - x\log\pi \end{aligned} \qquad \text{(C.2)}$$

As pointed before , taking derivatives on both sides of (C.2) we could assign a finite value to the divergent sum $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \cos(x\log n)$, using a test function we could compute the singular integral $\int_{-\infty}^{\infty} dx \frac{df}{dx} \mathrm{Im}\left\{ \log \zeta\left(\frac{1}{2}+ix\right)\right\}$ with Poles being the non-trivial zeros over the line Re(s)=1/2 , is related to series $\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} f(\log n)$.

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