On the ‘Size’ of Einstein’s Spherically Symmetric Universe

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It is alleged by the Standard Cosmological Model that Einstein’s Universe is finite but unbounded. Although this is a longstanding and widespread allegation, it is nonetheless false. It is also alleged by this Model that the Universe is expanding and that it began with a Big Bang. These are also longstanding and widespread claims that are demonstrably false. The expanding, finite, unbounded Universe is inconsistent with General Relativity and is therefore false.

1. Introduction

A 3-D spherically symmetric metric manifold has, in the spherical-polar coordinates, the following form (see Appendix),

\[ ds^2 = B(R_c) dR_c^2 + R_c^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \] (1)

where \( B(R_c) \) and \( R_c = R_c(r) \) are a priori unknown analytic functions of the variable \( r \) of the simple line element

\[ ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \] (2)

\[ 0 \leq r \leq \infty. \]

Line elements (1) and (2) have precisely the same fundamental geometric form and so the geometric relations between the components of the metric tensor are exactly the same in each line element. The quantity \( R_c \) appearing in (1) is not the geodesic radial distance associated with the manifold it describes. It is in fact the radius of curvature, in that it determines the Gaussian curvature \( G = 1/R_c^2 \) (see Appendix). The geodesic radial distance distance, \( R_p \), from an arbitrary point in the manifold described by (1) is an intrinsic geometric property of the line element, and is given by

\[ R_p = \int \sqrt{B(R_c)} \, dR_c + C = \int \frac{dR_c}{\sqrt{B(R_c)}} \, dr + C, \]

where \( C \) is a constant of integration to be determined (see Appendix). Therefore, (1) can be written as

\[ ds^2 = dr_p^2 + R_c^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \]

where

\[ dr_p = \sqrt{B(R_c)} dR_c, \]

and

\[ 0 \leq r_p < \infty, \]

with the possibility of the line element being singular (undefined) at \( r_p = 0 \), since \( B(R_c) \) and \( R_c = R_c(r) \) are a priori unknown analytic functions of the variable \( r \). In the case of (2),

\[ R_c(r) \equiv r, \quad dr_p \equiv dr, \quad B(R_c(r)) \equiv 1, \]

from which it follows that \( R_c \equiv r_p \equiv r \) in the case of (2). Thus \( R_c \equiv r_p \) is not general, and only occurs in the special case of (2), which describes an Efkleethean* space.

The volume \( V \) of (1), and therefore of (2), is

\[ V = \int_0^{R_p} \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\varphi \]

\[ = 4\pi \int_{R_c(0)}^{R_c(r)} R_c^2(r) \sqrt{B(R_c(r))} \, dR_c(r), \]

\[ = 4\pi \int_0^r R_c^2(r) \sqrt{B(R_c(r))} \, \frac{dR_c(r)}{dr} \, dr, \]

although, in the general case (1), owing to the a priori unknown functions \( B(R_c(r)) \) and \( R_c(r) \), the line element (1) may be undefined at \( R_p(R_c(0)) = R_c(r = 0) = 0 \), which is the location of the centre of spherical symmetry of the manifold of (1) at an arbitrary point in the manifold. Also, since \( R_c(r) \) is a priori unknown, the value of \( R_c(0) \) is unknown and so it cannot be assumed that \( R_c(0) = 0 \). In the special case of (2), both \( B(R_c(r)) \) and \( R_c(r) \) are known.

Similarly, the surface area \( S \) of (1), and hence of (2), is given by the general expression,

\[ S = R_c^2(r) \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi R_c^2(r). \]

*For the geometry due to Efkleethees, usually and abominably rendered as Euclid.
This might not ever be zero, since, once again, \(R_\epsilon(r)\) is an a priori unknown function and so \(R_\epsilon(0)\) might not be zero. It all depends on the explicit form for \(R_\epsilon(r)\), if it can be determined in a given situation, and on associated boundary conditions. The Appendix to this paper describes the mathematics in more detail.

2. The radius of Einstein’s universe

Since a geometry is entirely determined by the form of its line element, everything must be determined from it. One cannot, as is usually done, merely foist assumptions upon it. The intrinsic geometry of the line element and the consequent geometrical relations between the components of the metric tensor determine all.

Consider the usual non-static cosmological line element

\[
\text{ds}^2 = dt^2 - \frac{e^{\varphi(t)}}{(1 + \frac{k}{4}r^2)^2} [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)],
\]

wherein it is usually simply assumed that \(0 \leq \bar{r} < \infty\). However, the range on \(\bar{r}\) must be determined, not assumed. It is easily proved that the foregoing usual assumption is false.

Once again note that in (3) the quantity \(\bar{r}\) is not a radial geodesic distance. In fact, it is not even a radius of curvature on (3). It is merely a parameter for the radius of curvature and the proper radius, both of which are well-defined by the form of the line element (describing a spherically symmetric metric manifold). The radius of curvature, \(R_\epsilon\), for (3)

\[
R_\epsilon = e^{\frac{1}{2}g(t)} \frac{\bar{r}}{1 + \frac{k}{4}r^2}.
\]

The proper radius for (3) is given by

\[
R_p = e^{\frac{1}{2}g(t)} \int_0^{\bar{r}} \frac{d\bar{r}}{1 + \frac{k}{4}r^2} = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left( \arctan \frac{\sqrt{k}}{2} \bar{r} + n\pi \right), \quad n = 0, 1, 2, ...
\]

Since \(R_p \geq 0\) by definition, \(R_p = 0\) is satisfied when \(\bar{r} = 0 = n\). So \(\bar{r} = 0\) is the lower bound on \(\bar{r}\). The upper bound on \(\bar{r}\) must now be ascertained from the line element and boundary conditions.

It is noted that the spatial component of (4) has a maximum of \(\frac{2}{\sqrt{k}}\) for any time \(t\), when \(\bar{r} = \frac{2}{\sqrt{k}}\). Thus, as \(\bar{r} \to \infty\), the spatial component of \(R_\epsilon\) runs from 0 to the maximum \(\frac{2}{\sqrt{k}}\), then back to zero. Also,

\[
\lim_{\bar{r} \to \infty} \frac{\bar{r}}{1 + \frac{k}{4}r^2} = 0.
\]

Transform (3) by setting

\[
R = R(\bar{r}) = \frac{\bar{r}}{1 + \frac{k}{4}r^2},
\]

which carries (3) into

\[
ds^2 = dt^2 - e^{\varphi(t)} \left[ \frac{dR^2}{1 - kR^2} + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (8)
\]

which has the form of expressions (A6) and (A7) (see Appendix).

The quantity \(R\) appearing in (8) is not a radial geodesic distance. It is only a radius of curvature in that it determines the Gaussian curvature \(G = \frac{1}{e^{\varphi(t)} R^2}\). The radius of curvature of (8) is

\[
R_\epsilon = e^{\frac{1}{2}g(t)} R,
\]

and the proper radius of Einstein’s universe is, according to (8),

\[
R_p = e^{\frac{1}{2}g(t)} \int_0^\infty \frac{dR}{\sqrt{1 - kR^2}} = \int_0^{\frac{\pi}{2}} \arcsin \frac{\sqrt{k}}{R} + 2m\pi, \quad m = 0, 1, 2, ...
\]

Now according to (7), the minimum value of \(R\) is \(R(\bar{r} = 0) = 0\). Also, according to (7), the maximum value of \(R\) is \(R(\bar{r} = \frac{2}{\sqrt{k}}) = \frac{1}{\sqrt{k}}\). \(R = \frac{1}{\sqrt{k}}\) makes (8) singular, although (3) is not singular at \(\bar{r} = \frac{2}{\sqrt{k}}\). Since by (7), \(\bar{r} \to \infty \Rightarrow R(\bar{r}) \to 0\), then if \(0 \leq \bar{r} < \infty\) on (3) it follows that the proper radius of Einstein’s universe is, according to (8),

\[
R_p = \int_0^\infty \frac{dR}{\sqrt{1 - kR^2}} \equiv 0.
\]

Therefore, \(0 \leq \bar{r} < \infty\) on (3) is false. Furthermore, since the proper radius of Einstein’s universe cannot be zero and cannot depend upon a set of coordinates (it must be an invariant), expressions (5) and (10) must agree. Similarly, the radius of curvature of Einstein’s universe must be an invariant (independent of a set of coordinates), so expressions (4) and (9) must also agree, in which case \(0 \leq R < \frac{1}{\sqrt{k}}\) and \(0 \leq \bar{r} < \frac{2}{\sqrt{k}}\). Then by (5), the proper radius of Einstein’s universe is

\[
R_p = e^{\frac{1}{2}g(t)} \int_0^{\frac{2}{\sqrt{k}}} \frac{d\bar{r}}{1 + \frac{k}{4}r^2} = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left[ \left( \frac{\pi}{4} + n\pi \right) - m\pi \right], \quad n, m = 0, 1, 2, ...
\]

\[
n \geq m.
\]
Setting $p = n - m$ gives for the proper radius of Einstein's universe,

$$R_p = \frac{2e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left( \frac{\pi}{4} + p\pi \right), \quad p = 0, 1, 2, ... \quad (12)$$

Now by (10), the proper radius of Einstein's universe is

$$R_p = e^{\frac{1}{2}g(t)} \int_0^{\frac{\pi}{2}} \frac{dR}{\sqrt{1 - kR^2}}$$

$$= e^{\frac{1}{2}g(t)} \left[ \left( \frac{\pi}{2} + 2n\pi \right) - m\pi \right], \quad n, m = 0, 1, 2, ...$$

Setting $q = 2n - m$ gives the proper radius of Einstein's universe as,

$$R_p = e^{\frac{1}{2}g(t)} \left( \frac{\pi}{2} + q\pi \right), \quad q = 0, 1, 2, ... \quad (13)$$

Then equating (12) with (13), it follows that these expressions are equal only when $p \to \infty$ and $q \to \infty$, in which case both (12) and (13) are infinite. Thus, the proper radius of Einstein's universe is infinite.

By (4), (7) and (9), the invariant radius of curvature of Einstein's universe is,

$$R_e \left( \frac{2}{\sqrt{k}} \right) = e^{\frac{1}{2}g(t)} \frac{\sqrt{k}}{\sqrt{R}} = 2$$

which varies with time.

3. The volume of Einstein's universe

The volume of Einstein's universe is, according to (3),

$$V = e^{\frac{1}{2}g(t)} \int_0^{\frac{\pi}{2}} \frac{\pi^2}{1 + \frac{1}{4}(k\pi^2)} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= \frac{8\pi e^{\frac{1}{2}g(t)}}{k^{\frac{3}{2}}} \left( \frac{\pi}{4} + p\pi \right), \quad p = 0, 1, 2, ... \quad (15)$$

The volume of Einstein's universe is, according to (8),

$$V = e^{\frac{1}{2}g(t)} \int_0^{\frac{1}{2}R^2} \frac{dR}{\sqrt{1 - kR^2}} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi$$

$$= e^{\frac{1}{2}g(t)} \frac{2\pi}{k^{\frac{3}{2}}} \left( \frac{\pi}{2} + (2n - m)\pi \right), \quad n, m = 0, 1, 2, ... $$

and setting $q = 2n - m$ this becomes,

$$V = \frac{2\pi e^{\frac{1}{2}g(t)}}{k^{\frac{3}{2}}} \left( \frac{\pi}{2} + q\pi \right), \quad q = 0, 1, 2, ... \quad (16)$$

Since the volume of Einstein's universe must be an invariant, expressions (15) and (16) must be equal. Equality can only occur when $p \to \infty$ and $q \to \infty$, in which case both (15) and (16) are infinite. Thus the volume of Einstein's universe is infinite.

In the usual treatment (8) is transformed by setting

$$R = \frac{1}{\sqrt{k}} \sin \chi, \quad (17)$$

to get

$$ds^2 = dt^2 - \frac{e^{\frac{1}{2}g(t)}}{k} \left[ d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (18)$$

where it is usually asserted, without any proof (see e.g. [1, 2, 3, 4]), that

$$0 \leq \chi \leq \pi \quad \text{(or } 0 \leq \chi \leq 2\pi), \quad (19)$$

and whereby (18) is not singular. However, the actual range on $\chi$ is, according to (7), (11), (12), and (13), $0 \leq \chi < \frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2, ...$, so that the radius of curvature of Einstein's universe is, by (18),

$$R_c = \frac{e^{\frac{1}{2}g(t)} \sin \chi}{\sqrt{k}}$$

which must be evaluated for $\chi = \frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2, ...$, giving

$$R_c = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}}$$

as the radius of curvature of Einstein's universe, in concordance with (4), (7), and (9). The proper radius of Einstein's universe is given by

$$R_p = \frac{1}{\sqrt{k}} \int_0^{\frac{\pi}{2} + 2n\pi} d\chi = \frac{e^{\frac{1}{2}g(t)}}{\sqrt{k}} \left( \frac{\pi}{2} + 2n\pi \right), \quad (20) \quad n = 0, 1, 2, ...$$

and since the proper radius of Einstein's universe is an invariant, (20) must equal (12) and (13). This can only occur for (20) when $n \to \infty$, and so Einstein's universe is infinite.

According to (18), the volume of Einstein's universe is,

$$V = e^{\frac{1}{2}g(t)} \int_0^{\frac{\pi}{2} + 2n\pi} \int_0^\pi \int_0^{2\pi} d\chi \sin \theta d\theta d\varphi$$

$$= 4\pi e^{\frac{1}{2}g(t)} \left( \frac{\pi}{2} + 2n\pi \right), \quad n = 0, 1, 2, ... \quad (21)$$

Since this volume must be an invariant, expression (21) must equal expressions (15) and (16). This can only occur for (21) when $n \to \infty$, in which case (21) is infinite, and so Einstein's universe has an infinite volume.
4. The ‘area’ of Einstein’s universe

Using (3), the invariant surface area of Einstein’s universe is

\[ S = e^{g(t)} R_c^2 \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\varphi = 4\pi R_c^2 e^{g(t)} \]

which must be evaluated for \( R_c(\bar{r} = \frac{2}{\sqrt{k}}) \), according to (4), and so

\[ S = \frac{4\pi e^{g(t)}}{k}. \]

By (8) the invariant surface area is

\[ S = e^{g(t)} R^2 \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\varphi = 4\pi R^2 e^{g(t)}, \]

which must, according to (7), be evaluated for \( R = \frac{1}{\sqrt{k}} \), to give

\[ S = \frac{4\pi e^{g(t)}}{k}. \]

By (18) the invariant surface area is

\[ S = \frac{e^{g(t)}}{k} \sin^2 \chi \int_0^\pi \int_0^{2\pi} \sin \theta d\theta d\varphi = \frac{4\pi e^{g(t)}}{k} \sin^2 \chi, \]

and this, according to (17), must be evaluated for \( \chi = \left(\frac{\pi}{2} + 2n\pi\right), n = 0, 1, 2, \ldots \), which gives

\[ S = \frac{4\pi e^{g(t)}}{k}. \]

Thus the invariant surface area of Einstein’s infinite universe depends on time and is finite at any particular time.

In similar fashion the invariant great ‘circumference’, \( C = 2\pi e^{\frac{1}{2}g(t)} R_c \), of Einstein’s universe is a function of time and is finite at any particular time, given by

\[ C = \frac{2\pi e^{\frac{1}{2}g(t)}}{\sqrt{k}}. \]

5. Generalisation of the line element

Line elements (3), (8) and (18) can be generalised in the following way. In (3), replace \( \bar{r} \) by \( |\bar{r} - \bar{r}_0| \) to get

\[ ds^2 = dt^2 - \frac{e^{g(t)}}{1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2} \left[ d\bar{r}^2 + |\bar{r} - \bar{r}_0|^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \]

where \( \bar{r}_0 \in \mathbb{R} \) is entirely arbitrary. Line element (22) is defined on

\[ 0 \leq |\bar{r} - \bar{r}_0| < \frac{2}{\sqrt{k}} \quad \forall \bar{r}_0, \]

i.e. on

\[ \bar{r}_0 - \frac{2}{\sqrt{k}} \leq \bar{r} < \frac{2}{\sqrt{k}} + \bar{r}_0 \quad \forall \bar{r}_0. \]  

This corresponds to \( 0 \leq R_c < \frac{1}{\sqrt{k}} \) irrespective of the value of \( \bar{r}_0 \), and amplifies the fact that \( \bar{r} \) is merely a parameter. Indeed, (4) is generalised to

\[ R_c = R_c(\bar{r}) = \frac{|\bar{r} - \bar{r}_0|}{1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2}, \]

where (23) applies. Note that \( \bar{r} \) can approach \( \bar{r}_0 \) from above or below. Thus, there is nothing special about \( \bar{r}_0 = 0 \). If \( \bar{r}_0 = 0 \) and \( \bar{r} \geq 0 \), then (3) is recovered as a special case, still subject of course to the range \( 0 \leq \bar{r} < \frac{2}{\sqrt{k}} \).

Expression (7) is generalised thus,

\[ |R - R_0| = \frac{|\bar{r} - \bar{r}_0|}{1 + \frac{k}{4}|\bar{r} - \bar{r}_0|^2}, \]

where \( R_0 \) is an entirely arbitrary real number, and so (8) becomes

\[ ds^2 = dt^2 - e^{g(t)} \left[ \frac{dR^2}{1 - k(R - R_0)^2} + |R - R_0|^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \]

where

\[ R_0 - \frac{1}{\sqrt{k}} \leq R < \frac{1}{\sqrt{k}} + R_0 \quad \forall R_0. \]  

Note that \( R \) can approach \( R_0 \) from above or below. There is nothing special about \( R_0 = 0 \). If \( R_0 = 0 \) and \( R \geq 0 \), then (8) is recovered as a special case, subject of course to the range \( 0 \leq R < \frac{1}{\sqrt{k}} \).

Similarly, (18) is generalised, according to (24), by setting

\[ |R - R_0| = \frac{1}{\sqrt{k}} \sin |\chi - \chi_0|, \]

where \( \chi_0 \) is an entirely arbitrary real number, and

\[ 0 \leq |\chi - \chi_0| < \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, \ldots \]

i.e.

\[ \chi_0 - \left(\frac{\pi}{2} + 2n\pi\right) \leq \chi < \left(\frac{\pi}{2} + 2n\pi\right) + \chi_0, \quad n = 0, 1, 2, \ldots \]

\[ \forall \chi_0 \in \mathbb{R}. \]

Note that \( \chi \) can approach \( \chi_0 \) from above or below. There is nothing special about \( \chi_0 = 0 \). If \( \chi_0 = 0 \) and \( \chi \geq 0 \), then (18) is recovered as a special case, subject of course to the range \( 0 \leq \chi < \frac{\pi}{2} + 2n\pi, \quad n = 0, 1, 2, \ldots \)
6. Recapitulation and conclusions

Einstein’s universe has an infinite volume and an infinite proper radius at any time, but has a finite surface area at any time, a finite radius of curvature (giving a finite Gaussian curvature) at any time, and a finite great circumference at any time. The surface area, Gaussian curvature, and great circumference are time dependent. This geometrical oddity is no more odd than the fact that Einstein’s indefinite metric admits of null vectors (i.e. non-zero vectors of zero length), zero volumes with finite non-negative surface areas [5, 6], that a vector parallelly transported around a closed loop does not point in the same direction when it returns to its starting point as it did at its starting point before parallel transport, and two different masses for a single gravitating body (the passive and the active mass) in an otherwise empty universe [7, 8], amongst other oddities. Whether or not such an odd geometry has any physical significance is not a problem with which I deal, but it is however the inescapable geometrical nature of Einstein’s universe when described by spherically symmetric metric manifolds.

The behaviour of the Gaussian curvature, the great circumference and the surface area of Einstein’s universe, are dependent upon the function \( e^{\kappa t} \). There is no way in which \( g(t) \) can be given any particular form, save for the introduction of some merely ad hoc arguments. This calls into question the legitimacy of the non-static line elements, for if \( g(t) \) is a constant function, there is no non-static spherically symmetric line element, and since \( g(t) \) cannot be determined in principle, the non-static line elements can have no meaning.

Appendix: Spherically Symmetric Metric Manifolds

Following the method suggested by Palatini, and developed by T. Levi-Civita [9], denote ordinary Ecfleethan 3-space by \( \mathbb{E}^3 \). Let \( \mathbb{M}^3 \) be a 3-dimensional metric manifold. Let there be a one-to-one correspondence between all points of \( \mathbb{E}^3 \) and \( \mathbb{M}^3 \). Let the point \( O \in \mathbb{E}^3 \) and the corresponding point in \( \mathbb{M}^3 \) be \( O' \). Then a point transformation \( T \) of \( \mathbb{E}^3 \) into itself gives rise to a corresponding point transformation of \( \mathbb{M}^3 \) into itself.

A rigid motion in a metric manifold is a motion that leaves the metric \( dl^2 \) unchanged. Thus, a rigid motion changes geodesics into geodesics. The metric manifold \( \mathbb{M}^3 \) possesses spherical symmetry around any one of its points \( O' \) if each of the \( \infty^3 \) rigid rotations in \( \mathbb{E}^3 \) around the corresponding arbitrary point \( O \) determines a rigid motion in \( \mathbb{M}^3 \).

The coefficients of \( dl'^2 \) of \( \mathbb{M}^3 \) constitute a metric tensor and are naturally assumed to be regular in the region around every point in \( \mathbb{M}^3 \), except possibly at an arbitrary point, the centre of spherical symmetry \( O' \in \mathbb{M}^3 \).

Let a ray \( i \) emanate from an arbitrary point \( O \in \mathbb{E}^3 \). There is then a corresponding geodesic \( i' \in \mathbb{M}^3 \) issuing from the corresponding point \( O' \in \mathbb{M}^3 \). Let \( P \) be any point on \( i \) other than \( O \). There corresponds a point \( P' \) on \( i' \in \mathbb{M}^3 \) different from \( O' \). Let \( g' \) be a geodesic in \( \mathbb{M}^3 \) that is tangential to \( i' \) at \( P' \).

Taking \( i \) as the axis of \( \infty^3 \) rotations in \( \mathbb{E}^3 \), there correspond \( \infty^3 \) rigid motions in \( \mathbb{M}^3 \) that leave only all the points on \( i' \) unchanged. If \( g' \) is distinct from \( i' \), then the \( \infty^3 \) rigid rotations in \( \mathbb{E}^3 \) about \( i \) would cause \( g' \) to occupy an infinity of positions in \( \mathbb{M}^3 \) wherein \( g' \) has for each position the property of being tangential to \( i' \) at \( P' \) in the same direction, which is impossible. Hence, \( g' \) coincides with \( i' \).

Thus, given a spherically symmetric surface \( \Sigma \in \mathbb{E}^3 \) with centre of symmetry at some arbitrary point \( O \in \mathbb{E}^3 \), there corresponds a spherically symmetric geodesic surface \( \Sigma' \in \mathbb{M}^3 \) with centre of symmetry at the corresponding point \( O' \in \mathbb{M}^3 \).

Let \( Q \) be a point in \( \Sigma \in \mathbb{E}^3 \) and \( Q' \) the corresponding point in \( \Sigma' \in \mathbb{M}^3 \). Let \( ds \) be a generic line element in \( \Sigma \) issuing from \( Q \). The corresponding generic line element \( ds' \in \Sigma' \) issues from the point \( Q' \). Let \( \Sigma \) be described in the usual spherical-polar coordinates \( r, \theta, \varphi \). Then

\[
\dot{d}s^2 = r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (A1)
\]

\[
r = |OQ|.
\]

Clearly, if \( r, \theta, \varphi \) are known, \( Q \) is determined and hence also \( Q' \in \Sigma' \). Therefore, \( \theta \) and \( \varphi \) can be considered to be curvilinear coordinates for \( Q' \in \Sigma' \) and the line element \( ds' \in \Sigma' \) will also be represented by a quadratic form similar to (A1). To determine \( ds' \), consider two elementary arcs of equal length, \( ds_1 \) and \( ds_2 \) in \( \Sigma \), drawn from the point \( Q \) in different directions. Then the homologous arcs in \( \Sigma' \) will be \( ds'_1 \) and \( ds'_2 \) drawn in different directions from the corresponding point \( Q' \). Now \( ds_1 \) and \( ds_2 \) can be obtained from one another by a rotation about the axis \( OQ \) in \( \mathbb{E}^3 \), and so \( ds'_1 \) and \( ds'_2 \) can be obtained from one another by a rigid motion in \( \mathbb{M}^3 \), and are therefore also of equal length, since the metric is unchanged by such a motion. It therefore follows that the ratio \( \frac{ds'_1}{ds} \) is the same for the two different directions irrespective of \( d\theta \) and \( d\varphi \), and so the foregoing ratio is a function of position, i.e. of \( r, \theta, \varphi \). But \( Q \) is an arbitrary point in \( \Sigma \), and so \( \frac{ds'_1}{ds} \) must have the same ratio for any corresponding points \( Q \) and \( Q' \). Therefore, \( \frac{ds'_1}{ds} \) is a function of \( r \) alone, thus

\[
\frac{ds'_1}{ds} = H(r),
\]

and so

\[
ds'^2 = H^2(r)dr^2 = H^2(r)r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (A2)
\]
where $H(r)$ is a priori unknown. For convenience set $R_c = R_c(r) = H(r)r$, so that (A2) becomes
\[ dσ^2 = R_c^2(\bar{\theta}^2 + \sin^2 \bar{θ}dϕ^2), \quad \text{(A3)} \]
where $R_c$ is a quantity associated with $M^3$. Comparing (A3) with (A1) it is apparent that $R_c$ is to be rightly interpreted in terms of the Gaussian curvature $K$ at the point $O$, i.e. in terms of the relation $K = \frac{1}{r^2}$ since the Gaussian curvature of (1) is $K = \frac{1}{r^2}$. This is an intrinsic property of all line elements of the form (A3) [9, 10]. Accordingly, $R_c$ can be regarded as a radius of curvature. Therefore, in (A1) the radius of curvature is $R_c = r$. Moreover, owing to spherical symmetry, all points in the corresponding surfaces $Σ$ and $Σ'$ have constant Gaussian curvature relevant to their respective manifolds and centres of symmetry, so that all points in the respective surfaces are unimilic.

Let the element of radial distance from $O \in E^3$ be $dr$. Clearly, the radial lines issuing from $O$ cut the surface $Σ$ orthogonally. Combining this with (A1) by the theorem of Pythagoras gives the line element in $E^3$
\[ dl^2 = dr^2 + r^2(\bar{θ}^2 + \sin^2 \bar{θ}dϕ^2), \quad \text{(A4)} \]
Let the corresponding radial geodesic from the point $O' \in M^3$ be $dg$. Clearly the radial geodesics issuing from $O'$ cut the geodesic surface $Σ'$ orthogonally. Combining this with (A3) by the theorem of Pythagoras gives the line element in $M^3$ as,
\[ dl'^2 = dg^2 + R_c^2(\bar{θ}^2 + \sin^2 \bar{θ}dϕ^2), \quad \text{(A5)} \]
where $dg$ is, by spherical symmetry, also a function only of $R_c$. Set $dg = \sqrt{B(r_c)}dR_c$, so that (A5) becomes
\[ dl'^2 = B(r_c)dr_c^2 + R_c^2(\bar{θ}^2 + \sin^2 \bar{θ}dϕ^2), \quad \text{(A6)} \]
where $B(r_c)$ is an a priori unknown function.

Setting $dR_p = \sqrt{B(r_c)}dR_c$ carries (A6) into
\[ dl'^2 = dP_c^2 + R_c^2(\bar{θ}^2 + \sin^2 \bar{θ}dϕ^2). \quad \text{(A7)} \]
Expression (A6) is the most general for a metric manifold $M^3$ having spherical symmetry about some arbitrary point $O' \in M^3$ [9, 11].

Considering (A4), the distance $R_p = |OQ|$ from the point at the centre of spherical symmetry $O$ to a point $Q \in Σ$, is given by
\[ R_p = \int_0^r dr = r = R_c. \]
Call $R_p$ the proper radius. Consequently, in the case of $E^3$, $R_p$ and $R_c$ are identical, and so the Gaussian curvature at any point in $E^3$ can be associated with $R_p$, the radial distance between the centre of spherical symmetry at the point $O \in E^3$ and the point $Q \in Σ$. Thus, in this case, $K = \frac{1}{R_p} = \frac{1}{R_c} = \frac{1}{r}$. However, this is not a general relation, since according to (A6) and (A7), in the case of $M^3$, the radial geodesic distance from the centre of spherical symmetry at the point $O' \in M^3$ is not given by the radius of curvature, but by
\[ R_p = \int_0^{R_p} dR_p = \int_{R_c(0)}^{R_c(r)} \sqrt{B(r_c)} \, dR_c(r) \]
\[ = \int_0^r \sqrt{B(r_c)} \frac{dR_c(r)}{dr} \, dr, \]
where $R_c(0)$ is a priori unknown owing to the fact that $R_c(r)$ is a priori unknown. One cannot simply assume that because $0 \leq r < \infty$ in (A4) it must follow that in (A6) and (A7) $0 \leq r < \infty$. In other words, one cannot simply assume that $R_c(0) = 0$. Furthermore, it is evident from (A6) and (A7) that $R_p$ determines the radial geodesic distance from the centre of spherical symmetry at the arbitrary point $O'$ in $M^3$ (and correspondingly so from $O$ in $E^3$) to another point in $M^3$. Clearly, $R_c$ does not in general render the radial geodesic length from the centre of spherical symmetry to some other point in a metric manifold. Only in the particular case of $E^3$ does $R_c$ render both the Gaussian curvature and the radial distance from the centre of spherical symmetry, owing to the fact that $R_p$ and $R_c$ are identical in that special case.

It should also be noted that in writing expressions (A4) and (A5) it is implicit that $O \in E^3$ is defined as being located at the origin of the coordinate system of (A4), i.e. $O$ is located where $r = 0$, and by correspondence $O'$ is defined as being located at the origin of the coordinate system of (A5), i.e. using (A7), $O' \in M^3$ is located where $R_p = 0$. Furthermore, since it is well known that a geometry is completely determined by the form of the line element describing it [1], expressions (A4) and (A6) share the very same fundamental geometry because they are line elements of the same form.

Expression (A6) plays an important rôlé in Einstein’s gravitational field.


