The Doppler Effect as It Is
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Abstract. We present a derivation of the formula that accounts for the Doppler shift in a monochromatic electromagnetic wave, free of the “plane wave” or “very high frequency assumption”. For the limit \( m = \frac{d}{\lambda} \to \infty \) (\( d \) source-receiver distance, \( \lambda \) wavelength in the rest frame of the source) we recover the accredited formula for the non-longitudinal Doppler shift.

1. Introduction
We first invoke the invariance of a plane monochromatic electromagnetic wave and performing the Lorentz transformation for the space-time coordinates introduced in order to define its phase from the inertial reference frame \( I \) to another inertial reference frame \( I' \), in the standard configuration with \( I \). Transformation equations are derived for the formulas that accounts for the aberration of light effect and for the Doppler Effect, in the case when the direction in which the wave crests propagate and the receiver moves, respectively, are different. [1],[2],[3]. The derivation leads to the accredited Doppler shift formula

\[
\nu = \nu' \frac{1 + \frac{V}{c} \cos \theta}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{1}
\]

where \( \nu \) and \( \nu' \) represent the frequencies of electromagnetic oscillations taking place in the same electromagnetic wave as detected from \( I \) and \( I' \) respectively. Here \( V \) represents the relative velocity of \( I' \) relative to \( I \).

Equation (1) could be derived without invoking the invariance of the phase.[4],[5].

In deriving (1) French [5] shows that it is the result of an assumption that “the distance traveled by the moving source between the emission of two successive wave crests is very much less” than the distance from the source to the receiver (i.e. if the source travels a very small distance during one cycle of its transmitter signal). Yngstrom [6] shows that in order to derive (1) we should make the assumption that the quantity

\[
\frac{V}{r} \frac{\Delta \nu'_s}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{2}
\]

should be very small, where \( \Delta \nu'_s \) stands for the proper period at which the source emits successive wave crests and \( r \) for the distance between source and receiver.
The purpose of our paper is to derive a Doppler shift formula free of assumptions concerning the magnitude of the physical quantities introduced in order to characterize the wave.

2. A free of assumptions non-longitudinal Doppler shift formula.
The scenario we follow is sketched in Figure 1. It involves a source of light \( S \) located at the origin \( O \) of its rest frame \( I \) and an observer \( R' \) located at the origin of time \( t=0 \) on the \( x \) axis at a distance \( d \) from the origin \( O \). \( R' \) moves with constant velocity \( V \) along a direction that makes an angle \( \theta \) with the positive direction of the \( x \) axis. \( R' \) receives, located there, a first wave crest emitted from \( O \) at a time \( -d/c \). (Figure 1.a) . Figure 1b depicts the situation when \( R' \) receives a second wave crest emitted from \( O \) at a time \( t=T_e \) and received by \( R' \) at a time \( T_r \).

Figure 1.a. A source of electromagnetic oscillations is located at the origin \( O \) of an inertial reference frame \( I \). A receiver \( R' \) is located at a distance \( d \) from \( S \) and moves with constant speed \( V \) along a direction that makes an angle \( \theta \) with the positive direction of the \( x \) axis. The origin of time \( (t=0) \) coincides with the moment when \( R' \) receives a first wave crest emitted by the source at a time \( -d/c \).

Figure 1b. Observer \( R' \) receives the second wave crest emitted at a time \( -d/c+T_e \).

We mark in Figure 1b we mark the distances traveled by the first wave crest, by the second wave crest and by the receiver \( R' \), during the reception of the two successive wave crests. Pythagoras’ theorem applied to Figure 1b leads to

\[
(d + VT_r \cos \theta)^2 + V^2 T_r^2 \sin^2 \theta = c^2 \left( T_r + \frac{d}{c} - T_e \right)^2.
\]

Solved for \( T_r \) (3) leads to

\[
\frac{T_r}{T_e} = \frac{A + \sqrt{A^2 - B}}{1 - \frac{V^2}{c^2}}
\]

where
\[ A = 1 - \left( \frac{d}{\lambda} \right) \left( 1 - \frac{V}{c} \cos \theta \right) \]  

(5)

and

\[ B = \left( 1 - \frac{V^2}{c^2} \right) \left( 1 - \frac{2D}{\lambda} \right) \]  

(6)

where \( \lambda = cT_e \) represents the wavelength of the electromagnetic wave in the rest frame of the source.

In the experiment described above the time interval \( T_e \) is measured as a difference between the readings of the same clock \( K_0 \) located at the origin \( O \) and, by definition represents a proper time interval (proper emission period). The time interval \( T_r \) (reception period) is measured as a difference between the reading of a clock of \( I \) located where the second wave crest is received and the reading of another clock of \( I \) located where the first wave crest is emitted. Using his wristwatch observer \( R' \) measures a proper time interval \( T'_r \) between the receptions of the two successive wave crests. This time interval is related to \( T_r \) by the time dilation formula

\[ T_r = \frac{T'_r}{\sqrt{1 - \frac{V^2}{c^2}}} \]  

(7)

with which (4) becomes

\[ \frac{T'_r}{T_r} = \frac{A + \sqrt{A^2 - B}}{\sqrt{1 - \frac{V^2}{c^2}}} = D \]  

(8)

defining a Doppler factor \( D \), characteristic for the followed scenario. Since it is the result of a scenario which involves a stationary source of successive wave crests and an observer who moves along a direction that does not coincide with the direction in which the wave crests move, Equation (8) accounts for the non-longitudinal Doppler shift free of assumptions concerning the magnitudes of the physical quantities it involves.

Equation (8) accounts for the Doppler shift being the result of a scenario which involves a stationary source of successive wave crests and an observer who moves along a direction that does not coincide with the direction in which the wave crests move, free of assumption concerning the magnitude of the involved physical quantities. The Doppler factor \( D \) depends on the dimensionless factor \( m = d/\lambda \), a quotient between the initial source-receiver distance \( d \) and \( \lambda = cT_e \) the wavelength in the rest frame of the source.
We present in Figure 2 the variation of the Doppler factor $D$ with the angle $\theta$ for a constant value of $V/c$ and different values of $m=d/\lambda$ as a parameter.

An inspection of Figures 2 reveals the fact that with increasing values of $m$ the Doppler factor $D$ tends to a limit suggesting that we should consider the limit of $D$ for $m \to \infty$. Doing so and taking into account that the angles $\theta$ and $\theta'$ are related by the aberration of light formula

$$\cos \theta = \frac{\cos \theta' + \frac{V}{c}}{1 + \frac{V}{c} \cos \theta'}$$ \quad (9)

the result is

$$D_{m \to \infty} = \frac{\sqrt{1 - \frac{V^2}{c^2}}}{\frac{V}{c} \cos \theta} = \frac{1 + \frac{V}{c} \cos \theta'}{\sqrt{1 - \frac{V^2}{c^2}}}.$$ \quad (10)

Thus we have recovered the Doppler shift formula obtained invoking the invariance of the phase of a plane wave.

Figure 2. The variation of the Doppler factor $D$ with the angle for a constant value of the relative speed $\beta=V/c=0.6$ and different values of the dimensionless parameter $m=d/\lambda$.

3. The “very small period” (“very high frequency”) assumption
The condition $m \to \infty$ could be achieved imposing the condition $\lambda \to 0$ ($v \to \infty$) and we say that the “very high frequency” assumption is made. The assumption leads to the accredited formula for the Doppler shift derived above. We present such a derivation.[7]
Consider that a point-like source of electromagnetic oscillations located at the origin O and at rest in I emit a short light signal along a direction that makes an angle $\theta$ with the positive direction of the x axis, when clock $K_0$ located at O reads $t_e$. After a given time of propagation the light signal arrives at point M($r, \theta$) when clock K located at that point reads $t_r$. The clocks K and $K_0$ being standard synchronized, readings $t_r$ and $t_e$ are related by

$$t_r = t_e + \frac{r}{c}. \quad (11)$$

Differentiating both sides of (11) leads to

$$dt_r = dt_e + \frac{dr}{c}. \quad (12)$$

Equation (12) holds for each value of $t_e$ and so even for $t_e=0$, a value in which we are interested in. Taking into account that by definition

$$\frac{dr}{dt_r} = \frac{V}{c} \cos \theta \quad (13)$$

represents the radial component of the instantaneous velocity of an observer of the I’ frame located at the point M($r, \theta$) equation (12) leads to

$$\frac{dt_e}{dt_r} = 1 - \frac{V}{c} \cos \theta. \quad (14)$$

We can consider that $dt_e$ and $dt_r$ represent the period at which the source emits two successive wave crests and the period at which they are received at point M respectively. Under such conditions $dt_e$ represents a proper time interval, $dt_r$ representing a coordinate time interval. The wrist watch of observer of I’ mentioned above, measures a proper time interval $dt'_r$ related to $dt_r$ by the time dilation formula

$$dt'_r = \frac{dt'_r}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (15)$$

with which (14) becomes

$$\frac{dt_e}{dt'_r} = \frac{1 - \frac{V}{c} \cos \theta}{\sqrt{1 - \frac{V^2}{c^2}}}. \quad (16)$$

Expressed as a function of frequencies ($\nu=1/dt_e$; $\nu'=1/dt'_r$) (16) takes its final shape

$$\nu' = \nu \frac{1 - \frac{V}{c} \cos \theta}{\sqrt{1 - \frac{V^2}{c^2}}}. \quad (17)$$

the accredited formula for the Doppler shift obtained making the “very small period assumption” small enough in order to ensure that the moving receiver receives two successive wave crests being located at the same point in space.
4. Conclusions

In the physics of wave propagation, a plane wave (also spelled planewave) is a constant-frequency wave whose wave fronts (surfaces of constant phase) are infinite parallel planes of constant amplitude normal to the phase velocity vector.

By extension, the term is also used to describe waves that are approximately plane waves in a localized region of space. For example, a localized source such as an antenna produces a field that is approximately a plane wave in its far-field region. Equivalently, the "rays" in the limit where ray optics is valid (i.e. for propagation in a homogeneous medium over length scales much longer than the wavelength) correspond locally to approximate plane waves.

The derivation of the Doppler shift formula presented above shows that the assumptions free formula leads to the formula derived invoking the invariance of the phase of a plane wave only for very high values of the dimensionless parameter \( m = \frac{d}{\lambda} = \frac{d}{cT} \), i.e. when the initial source-receiver distance is very high. The parameter \( m \) suggests considering that the very high values of it could be obtained making the "very small wave length" i.e. “very high frequency” assumptions. As we see our \( m \) contains all the elements of the condition imposed in [6]. Making that assumption equation (10) could be derived, presenting it expressed via infinitesimal \( dt_e \) and \( dt_r \). [7],[8]. The longitudinal Doppler shift formula (\( \theta = 0 \) and \( \theta = \pi \)) os m independent.

References


