

There is always a prime between n^2 and $(n + 1)^2$

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Abstract I give an answer in the affirmative to the following

unanswered question :

Is there always a prime between n^2 and $(n + 1)^2$

where n is a natural number . The question

represents a famous unsolved problem in

Mathematics . I employ some familiar ideas in

Number Theory.

Seeking for a contradiction , I assume that there exists a natural number n , such that there exist no prime numbers between n^2 and $(n + 1)^2$

I apply the well known theorem:

There is a constant A such that :

$$(1) \sum_{p \leq x} \frac{1}{p} = \log \log x + A + O\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2$$

O is the big oh notation . The Summation is taken over all primes p less than or equal to x .

According to formula (1)

$$(2) \sum_{p \leq n^2} \frac{1}{p} = \log \log n^2 + A + O\left(\frac{1}{\log n^2}\right)$$

$$(3) \sum_{p \leq (n+1)^2} \frac{1}{p} = \log \log ((n+1)^2) + A + O\left(\frac{1}{\log(n+1)^2}\right)$$

Since I have assumed that there exist no prime numbers n^2 and $(n+1)^2$, we deduce that the left – hand sides of (2) and (3) are equal , hence

$$(4) \log \log n^2 + A + O\left(\frac{1}{\log n^2}\right) = \log \log ((n+1)^2) + A + O\left(\frac{1}{\log(n+1)^2}\right)$$

Hence we get

$$(5) \log \log ((n+1)^2) - \log \log n^2 = O\left(\frac{1}{\log n^2}\right) - O\left(\frac{1}{\log(n+1)^2}\right)$$

We know that

$$(6) \frac{\Pi(X)}{X} = O\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2$$

$\Pi(X)$ is the number of primes less than or equal to x

Now I assume that $(n+1)$ is prime .

Thus

$$(7) \Pi(n+1) = \Pi(n) + 1$$

We have that

$$(8) \frac{\Pi(n^2)}{n^2} = O\left(\frac{1}{\log n^2}\right) = O\left(\frac{1}{2 \log n}\right) = O\left(\frac{1}{\log n}\right)$$

$$(9) \frac{\Pi((n+1)^2)}{(n+1)^2} = O\left(\frac{1}{\log(n+1)^2}\right) = O\left(\frac{1}{2\log(n+1)}\right) = O\left(\frac{1}{\log(n+1)}\right)$$

Hence equation (5) becomes

$$(10) \log \log ((n+1)^2) - \log \log n^2 = O\left(\frac{1}{\log n}\right) - O\left(\frac{1}{\log(n+1)}\right)$$

We know that, since \log is the natural logarithm, and since the function $\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right)$ is a decreasing function,

$$(11) \left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right) < \frac{1}{2(n+1)} \text{ for all } n > 10$$

We know also that

$$(12) O\left(\frac{1}{\log n}\right) = \frac{\Pi(n)}{n}$$

$$(13) O\left(\frac{1}{\log(n+1)}\right) = \frac{\Pi(n+1)}{(n+1)}$$

Now using (11), (12), and (13), we can rewrite equation (10) in the form

$$(14) \log \log ((n+1)^2) - \log \log n^2 = \frac{\Pi(n)}{n} - \frac{\Pi(n+1)}{(n+1)} + t \text{ (for } n > 10)$$

where t is a small number ($t < \frac{1}{2(n+1)}$)

If we substitute from equation (7) we get

$$(15) \log \log ((n+1)^2) - \log \log n^2 = \frac{\Pi(n)}{n} - \frac{\Pi(n)+1}{(n+1)} + t$$

$$= \frac{\Pi(n)}{n(n+1)} - \frac{1}{(n+1)} + t$$

That is

$$(16) \log \log ((n+1)^2) - \log \log n^2 = \frac{1}{(n+1)} \left(\frac{\Pi(n)}{n} - 1 + t(n+1) \right)$$

Since

$$(17) \frac{\Pi(n)}{n} < 0.5 \text{ for } n > 10$$

and

$$(18) t(n+1) < \frac{1}{2(n+1)} \times (n+1) = 0.5$$

we conclude that the right – hand side of equation (16) is negative, and its left – hand side is positive. Hence we arrive at a contradiction.

Now I assume that $(n+1)$ is a composite number. Let r be the greatest prime less than or equal to n .

According to a theorem in Number Theory, if k is a natural number, then there is a prime between k and $2k$.

Hence if n is even we have :

$$(19) \frac{n}{2} \leq r < n$$

and if n is odd

$$(20) \frac{n-1}{2} \leq r \leq n$$

In both cases we have

$$(21) O\left(\frac{1}{\log r}\right) = O\left(\frac{1}{\log(n+1)}\right)$$

$$(22) O\left(\frac{1}{\log(r-1)}\right) = O\left(\frac{1}{\log n}\right)$$

Substituting in equation (10) we get

$$(23) \log \log ((n+1)^2) - \log \log n^2 = O\left(\frac{1}{\log(r-1)}\right) - O\left(\frac{1}{\log r}\right)$$

We know that

$$(24) \frac{\Pi(r)}{r} = O\left(\frac{1}{\log r}\right)$$

$$(25) \frac{\Pi(r-1)}{(r-1)} = O\left(\frac{1}{\log(r-1)}\right)$$

We know that, since \log is the natural logarithm, and since the function $\left(\frac{1}{\log(r-1)} - \frac{1}{\log r}\right)$ is a decreasing function,

$$(26) \left(\frac{1}{\log(r-1)} - \frac{1}{\log r}\right) < \frac{1}{2r} \text{ for all } r > 10$$

Now using (24), (25), and (26), we can rewrite equation (23) in the form

$$(27) \log \log ((n+1)^2) - \log \log n^2 = \frac{\Pi(r-1)}{(r-1)} - \frac{\Pi(r)}{r} + t \text{ (for } r > 10)$$

where t is a small number ($t < \frac{1}{2r}$)

But we know that

$$(28) \pi(r) = \pi(r-1) + 1$$

Hence equation (27) takes the form

$$(29) \log \log ((n+1)^2) - \log \log n^2 = \frac{\pi(r-1)}{(r-1)} - \frac{\pi(r-1)+1}{r} + t$$

$$= \frac{\pi(r-1)}{r(r-1)} - \frac{1}{r} + t$$

$$= \frac{1}{r} \left(\frac{\pi(r-1)}{(r-1)} - 1 + tr \right)$$

That is

$$(30) \log \log ((n+1)^2) - \log \log n^2 = \frac{1}{r} \left(\frac{\pi(r-1)}{(r-1)} - 1 + tr \right)$$

We know that

$$(31) \frac{\pi(r-1)}{(r-1)} < 0.5 \quad \text{for } r > 11$$

and

$$(32) t < \frac{1}{2r}$$

That is

$$(33) tr < 0.5$$

Hence we conclude that the right – hand side of equation (30) is negative , and its left – hand side is positive . This is a contradiction.

This ends my proof .

References

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